석 사 학 위 논 문
Master's Thesis

# 완전 유계 거리공간의 표상 및 범주론적 정립

# Representations of Totally Bounded Metric Spaces and Their Categorical Formulation

2019

임 동 현 (林 東 炫 Lim, Donghyun)

한 국 과 학 기 술 원

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임 동 현

위 논문은 한국과학기술원 석사학위논문으로
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# Representations of Totally Bounded Metric Spaces and Their Categorical Formulation

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Advisor: Martin Ziegler

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partial fulfillment of the requirements for the degree of
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Approved by

Martin Ziegler
Professor of School of Computing

The study was conducted in accordance with Code of Research Ethics[1].

[1] Declaration of Ethical Conduct in Research: I, as a graduate student of Korea Advanced Institute of Science and Technology, hereby declare that I have not committed any act that may damage the credibility of my research. This includes, but is not limited to, falsification, thesis written by someone else, distortion of research findings, and plagiarism. I confirm that my thesis contains honest conclusions based on my own careful research under the guidance of my advisor.

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**초 록**

유형-2 계산이론의 틀에서, 연속 공간의 표상은 문제의 계산가능성과 계산복잡도에 큰 영향을 끼친다. 본 논문에서는 좋은 표상의 기준으로서 "정량적 허용가능성"을 제시한다. 이는 크라이츠와 바이라흐가 1985 년에 발표한 고전적 결과를 정제한 것이다. 제2가산 $T_0$ 공간을 완전 유계 거리공간으로 구체화한다. 공간의 표상이 정량적으로 허용가능할 경우 함수의 연속성의 모듈러스와 실현체의 연속성의 모듈러스 사이에 밀접한 관계가 있음을 보인다. 표상공간을 범주로서 정립하고 그 범주가 모든 유한 곱을 가짐을 보인다.

**핵 심 낱 말** 계산 해석학, 유형-2 계산이론, 허용가능성, 완전 유계 거리공간, 연속성의 모듈러스, 범주론

**Abstract**

In the framework of Type-2 Theory of Effectivity, representations of continuous spaces affect computability and computational complexity of problems drastically. We propose "quantitative admissibility" as a criterion for sensible representations. Quantitative admissibility is a refinement of classical admissibility notion by Kreitz and Weihrauch, 1985. Classical setting of second-countable $T_0$ spaces is concretized to totally bounded metric spaces. We show that there is a close correspondence between modulus of continuity of a function and that of its realizer when the representations are quantitatively admissible. We formulate the represented spaces as categories and show that they have all finite products.

**Keywords** Computable Analysis, Type-2 Theory of Effectivity, Admissibility, Totally Bounded Metric Space, Modulus of Continuity, Category Theory

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# Mathematical Symbols

| | |
|---|---|
| $\mathbb{N}$ | $\{0, 1, 2, \cdots\}$ |
| $\Sigma$ | two-element alphabet set $\{\texttt{0}, \texttt{1}\}$ |
| $\Sigma^n$ | the set of all strings of length $n$ over $\Sigma$ |
| $\Sigma^*$ | the set of all finite-length strings over $\Sigma$ |
| $\Sigma^\omega$ | the Cantor space, consisting of all infinite-length strings over $\Sigma$ equipped with metric $d(p, q) = 2^{-\min\{i \mid p_i \neq q_i\}}$ |
| $uv$ | the concatenation of $u \in \Sigma^*$ and $v \in \Sigma^* \cup \Sigma^\omega$ |
| $u\Sigma^\omega$ | $\{uv \mid v \in \Sigma^\omega\}$ |
| $u_i$ | the $i$th character of $u \in \Sigma^* \cup \Sigma^\omega$; 0-indexed |
| $\iota$ | the embedding map $u_0 u_1 \cdots u_{n-1} \mapsto \texttt{110}u_0\texttt{0}u_1\texttt{0}\cdots\texttt{0}u_{n-1}\texttt{011} : \Sigma^* \to \Sigma^*$ |
| $u \sqsubseteq v$ | $u \in \Sigma^*$ is a prefix of $v \in \Sigma^* \cup \Sigma^\omega$. |
| $u \lhd v$ | $u \in \Sigma^*$ is a consecutive substring of $v \in \Sigma^* \cup \Sigma^\omega$. |
| $p_{<n}$ | $p_0 p_1 \cdots p_{n-1}$, the prefix of $p \in \Sigma^* \cup \Sigma^\omega$ of length $n$ |
| $p[i \cdots j)$ | $p_i p_{i+1} \cdots p_{j-1}$ |
| $\langle -, - \rangle$ | the bijective Cantor paring function $\langle a, b \rangle = \frac{(a+b)(a+b+1)}{2} + b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ |
| $\overline{B}(x, \epsilon)$ | closed ball centered at $x$ with radius $\epsilon$, $\{y \mid d(x, y) \leq \epsilon\}$ |
| $f :\subseteq X \to Y$ | a partial map from set $X$ to set $Y$ |
| id | the identity map |
| $\mu^{\underline{-1}}$ | the pseudoinverse $\mu^{\underline{-1}}(n) = \min\{i \mid \mu(i) \geq n\}$ |

Table 1: Mathematical Symbols Used Throughout Thesis

# Chapter 1. Introduction

## 1.1 Motivation

Some problems are computable, while some not. A large number of problems are computable. Examples include boolean satisfiability, sorting, finding the shortest path, primality testing, prime factorization, string matching, etc. We already have our state-of-the-art modern digital technology being able to implement Turing-complete computers. We can compute them all, and we are happy!

Or are we? It is certainly not the end of the story. Each of those computable problems has different *complexity*: primarily the time and the space required to compute it. Those problems, while all being computable, are not understood the same. They are magnified and examined more closely. They are categorized into finer classes beyond just *being computable*; for example, from $\mathcal{P}$, $\mathcal{NP}$, $\mathcal{EXPTIME}$, $\mathcal{PSPACE}$, $\mathcal{NPSPACE}$, to $\mathcal{EXPSPACE}$. Theory of computational complexity studies the differences among already computable problems.

Kreitz and Weihrauch, in 1985, proved a classical result, so-called *The Main Theorem of Computable Analysis* [Kreitz and Weihrauch, 1985, Weihrauch, 2000]: a function $f : X \to Y$ is continuous if and only if $f$ is computable with an oracle, where $X$ and $Y$ are second-countable and $T_0$ topological spaces. The theorem is about continuity and computability. These are *qualitative* properties: either $f$ has the properties, or not at all. There is no middle.

As in the case of the theory of computational complexity, we wish to look more closely into those functions. Our work refines this classical result by Kreitz and Weihrauch [Kreitz and Weihrauch, 1985, Weihrauch, 2000]. We develop a theory that could say something *quantitative* about functions $f : X \to Y$.

## 1.2 Outline

**Chapter 1.** Except for the table of contents and other miscellaneous things that precede, the thesis starts with Chapter 1, which is what you are looking at now. As you see, needless to say, Chapter 1 tries to introduce the whole thesis.

**Chapter 2.** Our interest lies in computation of functions $X \to Y$, where $X$ and $Y$ are mathematical spaces. Common examples of domain and codomain include $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. A Turing machine, the most common model of computation, does not work on various mathematical objects directly; it works only on strings. Mathematical objects have to be encoded into strings. Chapter 2 formalizes the notion of encoding and defines computation of functions $X \to Y$, where $X$ and $Y$ are abstract mathematical spaces. The materials in this chapter are from literatures on computable analysis, such as [Brattka et al., 2008] or [Weihrauch, 2000].

**Chapter 3.** Mathematical spaces cannot be encoded arbitrarily; there are proper ways of encoding. One such proper encoding is constructed, called *standard representations* [Kreitz and Weihrauch, 1985]. A criterion of *admissibility* is introduced, which all sensible encodings have to satisfy. The main theorem of computable analysis, that a function is continuous if and only if it admits a continuous realizer, is described. The theorem depends heavily on the assumption of admissibility.

**Chapter 4.** This chapter is a quantitative analogue of Chapter 3. Chapter 3 is about second-countable and $T_0$ topological spaces. They are refined to totally bounded metric spaces. *Modulus of continuity* is introduced as a quantitative analogue of continuity; *Kolmogorov entorpy* [Kolmogorov and Tikhomirov, 1959] is introduced as a quantitative analogue of second-countability. *Standard representations* for totally bounded metric spaces are constructed. *Quantitative admissibility* for encodings of totally bounded metric spaces is defined. *The Quantitative Main Theorem* is stated and proved, that asserts that there is a close correspondence between a function and its realizer in terms of modulus of continuity.

**Chapter 5.** This chapter puts the theory developed in previous chapters into the framework of category theory. Specific formulations are different from common approaches from the literature [Pauly, 2015, Pauly and Brecht, 2015]. Two categories are introduced. One is a category **RTop** of represented second-countable $T_0$ spaces. The other is a category **RMet** of represented totally bounded metric spaces. Objects are pairs $(X, \delta)$ of a space $X$ and an admissible representation $\delta$. Arrows are pairs $(f, F)$ of a function $f$ and its realizer $F$. We show that **RTop** and **RMet** have all finite products.

**Chapter 6.** This chapter concludes the thesis with a discussion on limitations and future work.

# Chapter 2. Type-2 Computation

## 2.1 Infinite Streams of Input and Output

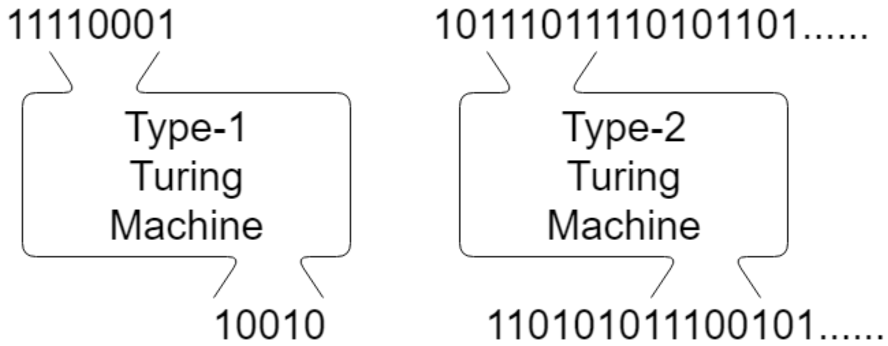



Figure 2.1: Type-1 and Type-2 Turing Machines

Type-1 computation refers to the usual Turing machine model of computation with finite control unit and unbounded memory. Many equivalent models have been discovered [Arora and Barak, 2009, Fernández, 2009], but we will confine ourselves to Turing machines. However, we will not be concerned about detailed formalism of Turing machines; they are treated informally.

Both the input and the output of a Turing machine consist of finite strings over a finite alphabet set. A computation is considered complete only when the machine eventually halts, however long it takes, and the string written on the output tape at the time of termination is considered as the output for a given input. If the machine never halts for a given input, then the output is undefined. A type-1 Turing machine $M$ computes a partial function $F_M :\subseteq \Sigma^* \to \Sigma^*$.

Type-2 computation refers to basically the same notion as Type-1 computation, with the same finite control unit and unbounded memory, except that the input and the output are infinite in length (but still over a finite alphabet set). The machine works on to write out symbols one by one, which, once written, never be altered. Only when the machine writes out infinitely many sympols is the input considered to have a valid output. Otherwise, in case the machine halts or stops writing output symbols, the output for the given input is undefined.

**Definition 1** (Type-2 Computability [Brattka et al., 2008])**.** A type-2 Turing machine $M$ *computes* a partial function $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ if for all $p \in \mathrm{dom}(F)$, $M$, when started with $p$ on its input tape, produces output $F(p)$. A partial function $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ is *computable* if there exists a type-2 Turing machine that computes $F$.

**Observation 2** (Composition of Computable Functions [Brattka et al., 2008])**.** If $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ and $G{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ are both computable, then so is $G \circ F$.

Definition 1 does not require anything on strings outside of $\mathrm{dom}(F)$. If $p \in \Sigma^\omega \setminus \mathrm{dom}(F)$, then $M(p)$ may or may not write infinitely many symbols. If $F$ is computable, then any restriction of $F$ on its domain is computable. The other option is to prohibit $M$ from printing infinitely many symbols on input $p \in \Sigma^\omega \setminus \mathrm{dom}(F)$. The difference is whether one's interest lies solely in machanical input-output translation itself, or in both the input-output translation and characterization of domain. One weakness of the other option is that even the parial identity functions $\mathrm{id}|_A :\subseteq \Sigma^\omega \to \Sigma^\omega$, with their domain restricted to some subsets $A \subseteq \Sigma^\omega$, are rendered uncomputable. There are only countably many turing machines, while there are $2^{2^\omega}$-many subsets $A$ of $\Sigma^\omega$. One condition under which the two possible definitions coincide is $\mathrm{dom}(F)$ being clopen, which is equivalent to $\mathrm{dom}(F) = W\Sigma^\omega$ for some finite subset $W \subseteq \Sigma^*$, which is again equivalent to decidability of $\mathrm{dom}(F)$ [Weihrauch, 2000].

We would like more partial functions to be computable, and are interested more in computation itself than characterization of domain, hence the choice of ignoreing input $p \in \Sigma^\omega \setminus dom(F)$. Our definition is in accordance with [Brattka et al., 2008]. However, two distinct definitions will make little difference to our main contribution, the quantitative main theorem, as the result is concerned about continuity rather than computability.

In case of type-1 computation, time cost of an algorithm is formalized as a partial function $t :\subseteq \mathbb{N} \to \mathbb{N}$, mapping $n$ to the maximum number of steps until termination among all input strings of length $n$ on which the machine halts. The same formalization cannot be applied for type-2 case as the machine never halts. Instead, the concept of time cost is defined as a partial function $t :\subseteq \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, mapping $n$ to the maximum number of steps until $n$th output symbol among all input strings on which the machine writes out infinitely many symbols. Note that $t$ may be $\infty$ as shown by the the following algorithm implemented by a type-2 Turing machine $M$:

**for** $i = 0$ *to* $\infty$ **do**
  **if** *ith input symbol is* `1` **then**
    print(`1`)
  **end**
**end**

Note that for $w = \mathtt{0}^n\mathtt{1}^\omega$, $M(w)$ takes $\Omega(n)$ steps until the 0th output. Also note that

$$\mathrm{dom}(f_M) = \{w \in \Sigma^\omega \mid w \text{ has infinitely many number of } \mathtt{1}\text{'s}\}$$

is not compact. In fact, $t(n)$ is always finite when the domain is compact.

## 2.2 Representations

The usual discrete theory of computation and computational complexity are concerned with computational problems such as: (1) Given a natural number $n$, what is the $n$th prime number? (2) Given a graph, does it have a Hamiltonian path? (3) Given a proportional logic formula, how many satisfying assignments are there? (4) Given a description of a program, does it halt or loop forever? These problems involve mathematical objects such as numbers, graphs, formulas, and programs. Turing machines, however, do not directly work on a variety of mathematical objects. They work only on strings. Here is where the idea of encodings comes in: giving names by strings over a finite alphabet set $\Sigma$. Let $X$ be a set of objects of our interest. $X$ could be encoded using a partial map $\delta_X :\subseteq \Sigma^* \to X$. It is surjective in that every object has at least one name and partial in that some names may be invalid. A mathematical object is now represented as a string of finite length. Let $Y$ be another set of objects of interest, having another encoding $\delta_Y :\subseteq \Sigma^* \to Y$. All previously mentioned problems can be modeled as a partial function $f :\subseteq X \to Y$, and its computation as a type-1 Turing machine $M$ computing a partial function $F_M :\subseteq \Sigma^* \to \Sigma^*$, so that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\uparrow & & {\scriptstyle \delta_Y}\uparrow \\ \Sigma^* & \xrightarrow{F} & \Sigma^* \end{array}.$$

This method of encoding by strings of finite length, however, has a critical limitation: $\Sigma^*$ is a countable set and there is no surjective map from $\Sigma^*$ to any set of continuum cardinality. Computation on $\mathbb{R}$, the set of real numbers, cannot be defined with this type-1 way of encoding. This issue can be handled by replacing $\Sigma^*$ with $\Sigma^\omega$, which contains infinitely many elements. We introduce a formal term *notation*/*representation* for encodings of sets of countable/continuum cardinality.

**Definition 3** (Notation [Weihrauch, 2000])**.** A *notation* of a set $X$ is a partial surjective map $\nu :\subseteq \Sigma^* \to X$.

**Definition 4** (Representation [Weihrauch, 2000])**.** A *representation* of a set $X$ is a partial surjective map $\delta :\subseteq \Sigma^\omega \to X$.

**Definition 5** (Realizer [Weihrauch, 2000])**.** Let $\delta_X :\subset \Sigma^\omega \to X$ and $\delta_Y :\subset \Sigma^\omega \to Y$ be representations of spaces $X$ and $Y$, respectively. Let $f :\subseteq X \to Y$ be a partial function. $F :\subset \Sigma^\omega \to \Sigma^\omega$ is a $(\delta_X, \delta_Y)$-*realizer* of $f$ if $f \circ \delta_X = \delta_Y \circ F$. We sometimes just say *realizer* when the representations are obvious from context.

The following diagram visually illustrates Definition 5 of realizer.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\uparrow & & {\scriptstyle \delta_Y}\uparrow \\ \Sigma^\omega & \xrightarrow{F} & \Sigma^\omega \end{array}$$

**Definition 6** (Computability [Weihrauch, 2000])**.** Let $X$ be a space and $\delta_X$ be its representation. Let $Y$ be a space and $\delta_Y$ be its representation. A partial function $f :\subseteq X \to Y$ is $(\delta_X, \delta_Y)$-*computable* if there exists a computable $(\delta_X, \delta_Y)$-realizer of $f$.

**Observation 7** (Composition [Weihrauch, 2000])**.** Let $\delta_X$, $\delta_Y$, and $\delta_Z$ be representations of $X$, $Y$, and $Z$, respectively. If $f :\subseteq X \to Y$ is $(\delta_X, \delta_Y)$-computable and $g :\subseteq Y \to Z$ is $(\delta_Y, \delta_Z)$-computable, then $g \circ f$ is $(\delta_X, \delta_Z)$-computable.

There are some subtleties worth reviewing. $f$, $F$, $\delta_X$, and $\delta_Y$ are all partial functions. Their compositions $f \circ \delta_X$ and $\delta_Y \circ F$ are understood as compositions of relations. Their equality $f \circ \delta_X = \delta_Y \circ F$ implies equality of domain and range as well.

Let $p \in \Sigma^\omega$. If $p \in \delta_X^{-1}[\mathrm{dom}(f)]$, then $F(p) \in \delta_Y^{-1}[\mathrm{range}(f)]$. In higher-level words, $F$ must map a valid name for an element in $\mathrm{dom}(f)$ to a valid name for an element in $\mathrm{range}(f)$. This is natural. On the other hand, if $p \notin \delta_X^{-1}[\mathrm{dom}(f)]$, then $F(p)$ must be either undefined or located outside of $\mathrm{dom}(\delta_Y)$. This technicality of partial function composition, however, raises no problem on computability of $F$. We can always take $F$ to be undefined on $\Sigma^\omega \setminus \delta_X^{-1}[\mathrm{dom}(f)]$. According to Definition 1, the machine $M$ computing $F$ need not care when $p \notin \delta_X^{-1}[\mathrm{dom}(f)]$ for input $p$. This convenience justifies the choice made in Definition 1.

Computability of a partial function $f :\subseteq X \to Y$ is defined with respect to two representations, $\delta_X :\subseteq \Sigma^\omega \to X$ for domain and $\delta_Y :\subseteq \Sigma^\omega \to Y$ for codomain. In type-1 theory of computation, the method of encoding mathematical objects mostly does not matter or is trivial. One can mostly convert back and forth between an adjacency matrix encoding and an adjacency list encoding of a graph, in asymptotically negligible amount of time. Nobody considers unary encoding of natural numbers to be a sensible one except a few special cases. In type-2 theory of computation, however, choice of representations matters in a critical way, as shown by Turing in 1937 [Turing, 1937].

**Example 8** (Incomputability of Multiplication by Three [Turing, 1937])**.** Consider a binary representation $\delta_{[0,1]} : \Sigma^\omega \to [0,1]$ defined by

$$w \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \cdot w_i,$$

and another binary representation $\delta_{[0,4]} : \Sigma^\omega \to [0,4]$ defined by

$$w \mapsto \sum_{i=0}^{\infty} 2^{-i+1} \cdot w_i.$$

Then the map $f : [0,1] \to [0,4]$ given by

$$f(x) = 3x$$

is not $(\delta_{[0,1]}, \delta_{[0,4]})$-computable.

*Proof.* Aiming for a contradiction, suppose that $f$ is $(\delta_{[0,1]}, \delta_{[0,4]})$-computable, by a type-2 Turing machine $M$. Feed $M$ with

$$w = \texttt{0101010101} \cdots .$$

Note that $\delta_{[0,1]}(w) = \frac{1}{3}$. Then $M$ must output either

$$u' = \texttt{001111111111} \cdots$$

or

$$u'' = \texttt{010000000000} \cdots .$$

Note that $\delta_{[0,4]}(u') = \delta_{[0,4]}(u'') = 1$. Let us first examine the case where $M$ outputs $u$. After finitely many number of steps, $M$ prints out $u'_1 = \texttt{0}$. Until this time, $M$ has seen only finitely many digits of $w$, say, until $n$th digit. Then $M$ would behave exactly the same until its 1st (not 0th) output symbol if the input to $M$ coincided with $w$ until $n$th digit. In particular, $M$ would output `00` as its 0th and 1st output symbols, given input

$$w' = \texttt{0101010101} \cdots \texttt{0101010111111111111} \cdots$$

that coincides with $w$ until $n$th digit and repeats `1` after that. $M$, however, must output some $v' \in \Sigma^\omega$ satisfying $\delta_{[0,4]}(v') = 3 \cdot \delta_{[0,1]}(w') > 1$. Then the first two digits of $v'$ cannot be `00`, reaching a contradiction. Now we examine the other case. Suppose that $M(w)$ outputs $u''$, whose first two digits are `01`. The reasoning is the same as in the previous case. This time we consider

$$w'' = \texttt{0101010101} \cdots \texttt{0101010100000000000} \cdots$$

that coincides with $w$ for enough number of digits and repeats `0` after that. Then $M(w'')$ must output some $v'' \in \Sigma^\omega$ satisfying $\delta_{[0,4]}(v'') = 3 \cdot \delta_{[0,1]}(w'') < 1$. The first two digits of $v''$ cannot be `01`, reaching a contradiction again. ☐

The argument for Example 8 could be understood intuitive and natural. Suppose that a machine is reading an input $0.33333333\cdots$, trying to compute its multiplication by 3. To determine the symbol for an output digit, the machine has to know whether the input is less than or greater than $\frac{1}{3}$. This is impossible since the machine always knows only finitely many digits of the input.

Addition of two real numbers is not computable either, with respect to binary representations. An informal proof of this fact is like this: Multiplication by two is trivially computable by simple shifting. Combining addition and multiplication by two, one can compute mulplication by three, if addition is computable.

A theory of computation on real numbers will not be so fruitful if it excludes such simple functions as addition or multiplication. We could overcome this problem by adopting different representations.

**Definition 9** (bin : $\mathbb{N} \to \Sigma^*$ [Kawamura et al., 2018])**.** Let us define a bijective map $\mathbb{N} \to \Sigma^*$ assigning a finite word over $\Sigma$ for each natural number. We can give a total order on $\Sigma^*$, first by length of the words, and then by the dictionary order on $\Sigma^n$ for each $n$. Let bin : $\mathbb{N} \to \Sigma^*$ be the order isomorphism.

Definition 9 is utilized to construct the dyadic representation in Definition 10. There is slight ambiguity regarding which digit (leftmost or rightmost) has more weight then the others. This ambiguity will make little difference for our purpose. Note that the map $n \mapsto |\mathrm{bin}(n)|$ grows logarithmically.

**Definition 10** (Dyadic Representation [Kawamura et al., 2018])**.** The *dyadic representation* is a partial surjective map $\delta{:}\subseteq \Sigma^\omega \to[0,1]$ defined by

$$\iota(\mathrm{bin}(a_0))\iota(\mathrm{bin}(a_1))\iota(\mathrm{bin}(a_2))\cdots\iota(\mathrm{bin}(a_n))\cdots \mapsto \lim_j a_j/2^j$$

where

$$\mathrm{dom}(\delta) = \{\iota(\mathrm{bin}(a_0))\cdots\iota(\mathrm{bin}(a_n))\cdots \in \Sigma^\omega \mid 0 \le a_n \le 2^n, |a_n/2^n - a_m/2^m| \le 2^{-n} + 2^{-m}\}.$$

Informally, the dyadic representation is a sequence of dyadic rationals between 0 and 1, with the precision increasing term by term, eventually converging to a single point in $\mathbb{R}$. Notice the additional condition $|a_n/2^n - a_m/2^m| \le 2^{-n} + 2^{-m}$ imposed on $\mathrm{dom}(\delta)$. This is justified by the fact that a sequence $(a_j/2^j)_j$ of dyadic rationals converges to a point $r \in R$ so that

$$|a_j/2^j - r| \le 2^{-j} \text{ for all } j \in \mathbb{N}$$

if and only if $(a_j/2^j)_j$ satisfies

$$|a_n/2^n - a_m/2^m| \le 2^{-n} + 2^{-m} \text{ for all } n, m \in \mathbb{N}.$$

**Definition 11** (Signed Binary Representation [Kawamura et al., 2018])**.** The *signed binary representation* is a partial surjective map $\sigma{:}\subseteq \Sigma^\omega \to[0,1]$ defined by

$$b \mapsto \frac{1}{2} + \sum_{m\ge 0}^{\infty} 2^{-m-2}\cdot(2b_{2m} + b_{2m+1} - 1)$$

where

$$\mathrm{dom}(\sigma) = \{00, 01, 10\}^\omega \subseteq \Sigma^\omega.$$

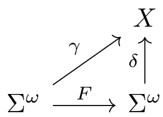
In Definition 10 and 11, the target set encoded is the unit interval $[0,1]$. Any closed interval of $\mathbb{R}$ can be encode similarly. We have chosen the unit interval for simplicity. It is possible to encode the whole

$\mathbb{R}$ using similar scheme, in *dyadic way* or *signed binary way*, by somehow handling the radix point. The representation for $\mathbb{R}$, however, has problems with computational complexity, rather than computability. There is no guarantee of precision however many digits have read by the machine. Note that $\mathbb{R}$ is not compact, which might be the cause of the problem.

Dyadic representation and signed binary representation do not suffer from the previous problem of binary representation. Addition, subtraction, multiplication and division are all computable with respect to both dyadic representation and signed binary representation [Brattka et al., 2008].

So, which representation to choose between dyadic representation and signed binary representation? In fact, they are equivalent in computability. Let us establish and formalize the notion of equivalence in computability by going through the idea of reduction.

**Definition 12** (Computable Reduction [Weihrauch, 2000])**.** Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to X$ be two partial functions. $\gamma$ *computably reduces* to $\delta$, or, put another way, $\gamma$ is *computably reducible* to $\delta$, if there exists a computable partial function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\gamma = \delta \circ F$.

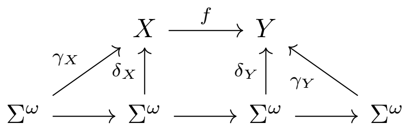



When $\gamma$ reduces to $\delta$, it is denoted simply as $\gamma \preceq \delta$. If both $\gamma \preceq \delta$ and $\delta \preceq \gamma$, than $\gamma$ and $\delta$ are said to be *computably equivalent* and denoted $\gamma \equiv \delta$.

**Observation 13** (Computable Reduction as a Relation [Weihrauch, 2000])**.** Computable reduction $\preceq$ is reflexive and transitive. Computable equivalence $\equiv$ forms an equivalence relation.

**Theorem 14** (Computable Reduction and Computability [Weihrauch, 2000])**.** *Let $f :\subseteq X \to Y$ be a partial function. Let $\gamma_X$ and $\delta_X$ be representations of $X$ such that $\gamma_X \preceq \delta_X$. Let $\gamma_Y$ and $\delta_Y$ be representations of $Y$ such that $\delta_X \preceq \gamma_Y$. If $f$ is $(\delta_X, \delta_Y)$-computable, then $f$ is $(\gamma_X, \gamma_Y)$-computable.*

*Proof.* Consider the following commutative diagram and reason about it.




□

**Corollary 15** (Computable Equivalence and Computability [Weihrauch, 2000])**.** *Let $f :\subseteq X \to Y$ be a partial function. Let $\gamma_X$ and $\delta_X$ be representations of $X$ such that $\gamma_X \equiv \delta_X$. Let $\gamma_Y$ and $\delta_Y$ be representations of $Y$ such that $\delta_X \equiv \gamma_Y$. $f$ is $(\delta_X, \delta_Y)$-computable if and only if $f$ is $(\gamma_X, \gamma_Y)$-computable.*

**Example 16** (Computable Equivalence of Dyadic and Signed Binary Representations)**.** Dyadic representation and signed binary representation are computably equivalent.

## 2.3 Computability and Continuity

Theorem 17 clearly shows the close relationship between computability and continuity. Simply put, every computable (with or without an oracle) partial function $\Sigma^\omega \to \Sigma^\omega$ is continuous since any finite prefix of an output depends solely on a finite prefix of the input. On the other hand, any continuous partial function $\Sigma^\omega \to \Sigma^\omega$ can be approximated by a monotone unbounded map $\Sigma^* \to \Sigma^*$, which we can take as an oracle.

**Theorem 17** (Oracle Computability and Continuity [Weihrauch, 2000])**.** *Let $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ be a partial function. $F$ is computable with an oracle $\Sigma^* \to \Sigma^*$ if and only if $F$ is continuous.*

*Proof.* ($\Rightarrow$) Let $F$ be computable by an oracle machine $M$. Let F(x) = y. Consider an open neighborhood $u\Sigma^\omega$ of $y$, where $u \in \Sigma^*$ is a finite prefix of $y$. At the time $M(x)$ writes $u$ on its output tape, it has seen only until a finite prefix $w$ of $x$. Then, $F(w\Sigma^\omega) \subseteq u\Sigma^\omega$.

($\Leftarrow$) Let $F$ be continuous. Define an oracle $\Omega : \Sigma^* \to \Sigma^*$ by

$$\Omega(w) := \begin{cases} \lambda & \text{if } F(w\Sigma^\omega) = \emptyset \\ \text{the longest prefix of } F(w\Sigma^\omega) \text{ of length at most } |w| & \text{otherwise} \end{cases}.$$

For any $p \in \mathrm{dom}(F)$, the sequence $(\Omega(p_{<n}))_{n\in\mathbb{N}} \subseteq \Sigma^*$ is monotone and unbounded, that is,

$$i \leq j \text{ implies } \Omega(p_{<i}) \sqsubseteq \Omega(p_{<j})$$

and

$$|\Omega(p_{<n})| \to \infty \text{ as } n \to \infty.$$

Note that unboundedness is from continuity of $F$: for any proper prefix $u \in \Sigma^*$ of $p$, there exists $w \in \Sigma^*$ such that $F(w\Sigma^\omega) \subseteq u\Sigma^\omega$. $\Omega(p_{<n})$ is always a prefix of $F(p)$ by definition of $f$. Thus,

$$\lim_{n\to\infty} \Omega(p_{<n}) = F(p).$$

A machine can utilize $\Omega$ to produce output $F(p)$ on input $p \in \mathrm{dom}(F)$. □

From now on we will be mainly concerned about continuity rather than computability. Theorem 17 works as a justification for this deviation to a seemingly unrelated concept.

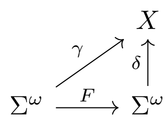
# Chapter 3. Qualitative Main Theorem

This chapter describes a classical result, so-called *The Main Theorem of Computable Analysis*, proved by Kreitz and Weirauch in 1985 [Kreitz and Weihrauch, 1985]. They showed that there is a close correspondence between a partial function $f :\subseteq X \to Y$ and its type-2 realizer $F{:\subseteq}\, \Sigma^\omega \to \Sigma^\omega$: $f$ is continuous if and only if there exists a continuous realizer $F$ of $f$. The assumptions of the theorem are weak, hence the generality. $X$ and $Y$ are assumed to be second-countable $T_0$ spaces, and each of the representations of $X$ and $Y$ must meet a criterion, called *admissibility*. We have seen from Example 8 that certain representations are bad. Admissibility constitutes a criterion to distinguish good representations from bad ones.

## 3.1 Continuous Reduction

The following definitions and theorems in this section, stated in terms of continuity but not computability, are analogous to those in Chapter 2.

**Definition 18** (Continuous Reduction [Weihrauch, 2000])**.** Let $\gamma{:\subseteq}\, \Sigma^\omega \to X$ and $\delta{:\subseteq}\, \Sigma^\omega \to X$ be two partial functions of $X$. $\gamma$ *continuously reduces* to $\delta$, or, put another way, $\gamma$ is *continuously reducible* to $\delta$, if there exists a continuous partial function $F{:\subseteq}\, \Sigma^\omega \to \Sigma^\omega$ such that $\gamma = \delta \circ F$.




When $\gamma$ reduces to $\delta$, it is denoted simply as $\gamma \preceq_t \delta$. If both $\gamma \preceq_t \delta$ and $\gamma \succeq_t \delta$, than $\gamma$ and $\delta$ are said to be *continuously equivalent* and denoted $\gamma \equiv_t \delta$.

Computational complexity theory also has the concept of *reduction*, a way of transforming instances of a problem to another problem, playing a crucial role in the development of the theory. In computational complexity theory, one's interest lies not just on a single problem, but rather a class of problems. For a class of $\mathcal{C}$ of problems, there are $\mathcal{C}$-complete problems, the problems to which any problem in $\mathcal{C}$ can be reduced. They form an equivalence class and are considered the most difficult problems contained in $\mathcal{C}$. Reduction between representations plays an analogous role for the development of the theory or representations. Representations that are maximal with respect to $\preceq_t$ have good mathematical and computational properties, as shown in this chapter.

**Observation 19** (Continuous Reduction as a Relation [Weihrauch, 2000])**.** Continuous reduction. $\preceq_t$ is reflexive and transitive. Continuous equivalence $\equiv_t$ forms an equivalence relation.

**Observation 20** (Continuous Reduction and Final Topology [Weihrauch, 2000])**.** Let $X$ be a space without topology. Let $\gamma{:\subseteq}\, \Sigma^\omega \to X$ and $\delta{:\subseteq}\, \Sigma^\omega \to X$ be representations of X such that $\gamma \preceq_t \delta$. Then

$$(\text{the final topology on } X \text{ with respect to } \gamma) \supseteq (\text{the final topology on } X \text{ with respect to } \delta).$$

**Observation 21** (Continuous Equivalence and Final Topology [Weihrauch, 2000])**.** Let $X$ be a space without topology. Let $\gamma{:\subseteq}\, \Sigma^\omega \to X$ and $\delta{:\subseteq}\, \Sigma^\omega \to X$ be representations of X such that $\gamma \equiv_t \delta$. Then

$$(\text{the final topology on } X \text{ with respect to } \gamma) = (\text{the final topology on } X \text{ with respect to } \delta).$$

**Theorem 22** (Continuous Reduction and Realizer [Weihrauch, 2000]). *Let $f :\subseteq X \to Y$ be a partial function. Let $\gamma_X$ and $\delta_X$ be representations of $X$ such that $\gamma_X \preceq_t \delta_X$. Let $\gamma_Y$ and $\delta_Y$ be representations of $Y$ such that $\delta_X \preceq_t \gamma_Y$. If $f$ admits a coutinuous $(\delta_X, \delta_Y)$-realizer, then $f$ also admits a continuous $(\gamma_X, \gamma_Y)$-realizer.*

*Proof.* Analogous to the proof of Theorem 14. □

**Corollary 23** (Continuous Equivalence and Realizer [Weihrauch, 2000]). *Let $f :\subseteq X \to Y$ be a partial function. Let $\gamma_X$ and $\delta_X$ be representations of $X$ such that $\gamma_X \equiv_t \delta_X$. Let $\gamma_Y$ and $\delta_Y$ be representations of $Y$ such that $\delta_X \equiv_t \gamma_Y$. Then $f$ admits a continuous $(\delta_X, \delta_Y)$-realizer if and only if $f$ admits a continuous $(\gamma_X, \gamma_Y)$-realizer.*

## 3.2 Standard Representations

**Definition 24** (Standard Representation [Kreitz and Weihrauch, 1985] [Weihrauch, 2000]). Let $X$ be a second-countable $T_0$ space. Let $\nu :\subseteq \Sigma^* \to \mathcal{B}$ be a notation for a countable base $\mathcal{B}$ of X. The *standard representation* $\delta :\subseteq \Sigma^\omega \to X$ with respect to $\nu$ is a representation of $X$ such that $\delta(p) = x$ if and only if

$$\{B \in \mathcal{B} \mid x \in B\} = \{\nu(w) \mid w \in \mathrm{dom}(\nu) \wedge \iota(w) \lhd p\}$$

where $\mathrm{dom}(\delta)$ is all those $p \in \Sigma^\omega$ that are mapped to an appropriate $x \in X$ by the above equation.

Informally, standard representation is defined so that, an enumeration of all base elements containing $x \in X$ constitutes a name for $x$. The order of enumeration does not matter and repetition is allowed. Separation axiom $T_0$ is assumed so that every point $x \in X$ has a different collection of base elements from all others. Second countability is assumed so that each base element has a finite-length name.

Every second-countable $T_0$ space has a standard representation. Note that we said *a* standard representation. There are infinitely many standard representations, since there are infinitely many notations $\Sigma^* \to \mathcal{B}$ of base $\mathcal{B}$.

**Lemma 25** (Properties of Standard Representation [Weihrauch, 2000]). *Let $\delta :\subseteq \Sigma^\omega \to X$ be a standard representation of $X$. Then,*

1. *$\delta$ is continuous.*
2. *$\delta$ is an open map.*
3. *the topology of $X$ is final with respect to $\delta$.*

*Proof.* Denote by $\nu :\subseteq \Sigma^* \to \mathcal{B}$ the notation of a base $\mathcal{B}$ from which $\delta$ is built.

1. Let $B \in \mathcal{B}$. Then,

$$\begin{aligned}\delta^{-1}[B] &= \{p \in \mathrm{dom}(\delta) \mid \iota(w) \lhd p \text{ for some } w \text{ with } \nu(w) = B\} \\ &= \bigcup_{u \in \Sigma^*,\, w : \nu(w) = B} u\,\iota(w)\,\Sigma^\omega\end{aligned}$$

   which is a union of open sets.

2. Let $u \in \Sigma^*$. Then,

$$\delta[u\Sigma^\omega] = \bigcap \{\nu(w) \mid w \in \mathrm{dom}(\nu), \iota(w) \lhd u\}$$

   which is a finite intersection of open sets.

3. It follows from item 1 and 2.

□

**Lemma 26** (Properties of Standard Representation [Weihrauch, 2000]). *Let $\delta :\subseteq \Sigma^\omega \to X$ be a standard representation of $X$. Then $\zeta \preceq_t \delta$ for every continuous partial map $\zeta :\subseteq \Sigma^\omega \to X$.*

$$\begin{array}{ccc} & & X \\ & \overset{\zeta}{\nearrow} & \uparrow \delta \\ \Sigma^\omega & \xrightarrow{F} & \Sigma^\omega \end{array}$$

*Proof.* Let $\nu :\subseteq \Sigma^* \to \mathcal{B}$ be the notation of the countable base $\mathcal{B}$ from that $\delta$ is built. Fix an enumeration $(w_i)_{i\in\mathbb{N}}$ of $\mathrm{dom}(\nu)$. Let $p \in \mathrm{dom}(\zeta)$. By continuity of $\zeta$, $\zeta(p) \in B$ if and only if $\zeta[p_{<n}\Sigma^\omega] \subseteq B$ for some $n \in \mathbb{N}$, where $B \in \mathcal{B}$. Define $h_p : \mathbb{N} \to \Sigma^*$ by

$$h\langle n, i\rangle = \begin{cases} \iota(w_i) & \text{if } \zeta[p_{<n}\Sigma^\omega] \subseteq \nu(w_i) \\ 1 & \text{otherwise} \end{cases}.$$

Define $F : \mathrm{dom}(\zeta) \to \Sigma^\omega$ by

$$F(p) = h_p(0)h_p(1)h_p(2)\cdots.$$

Let $p \in \mathrm{dom}(\zeta)$. $F(p)$ is a list of all and only $w_i$ with $\zeta(p) \in \nu(w_i)$, hence $\zeta = \delta \circ F$. $F$ is continuous since every finite prefix of $F(p)$ depends on a finite prefix of $p$. □

## 3.3 Admissibility

**Definition 27** (Admissibility [Kreitz and Weihrauch, 1985] [Weihrauch, 2000]). Let $X$ be a second-countable $T_0$ space. A representation $\delta$ of $X$ is *admissible* if

- $\delta$ is continuous and
- $\zeta \preceq_t \delta$ for every continuous partial map $\zeta :\subseteq \Sigma^\omega \to X$.

$$\begin{array}{ccc} & & X \\ & \overset{\zeta}{\nearrow} & \uparrow \delta \\ \Sigma^\omega & \xrightarrow{F} & \Sigma^\omega \end{array}$$

Admissibility is just another way of characterizing the class of representations continuously equivalent to a standard representation.

**Theorem 28** (Equivalent Condition of Admissibility [Weihrauch, 2000]). *Let $X$ be a second-countable $T_0$ space. Let $\delta$ be a representation of $X$. Then $\delta$ is admissible if and only if $\delta$ is continuously equivalent to a standard representation of $X$.*

*Proof.* ($\Rightarrow$) Apply Definition 27, Lemma 25, and Lemma 26.

($\Leftarrow$) Apply the fact that composition of two continuous partial functions is continuous, Lemma 26, and Observation 19. □

**Lemma 29** (Admissible Representation and Final Topology [Weihrauch, 2000]). *Let $\delta :\subseteq \Sigma^\omega \to X$ be an admissible representation of second-countable $T_0$ space $X$. Then the topology on $X$ is final with respect to $\delta$.*

*Proof.* Apply Observation 21 and Theorem 28. ☐

**Lemma 30** (Final Topology and Continuity)**.** *Let $A, X, Y$ be topological spaces. Let $\alpha :\subseteq A \to X$ and $f :\subseteq X \to Y$ be partial functions. Let the topology on $X$ be final with respect to $\alpha$. If $f \circ \alpha$ is continuous, then so is $f$.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha}\uparrow & \nearrow_{f\circ\alpha} & \\ A & & \end{array}$$

*Proof.* Assume that $f \circ \alpha$ is continuous. Let $U \subseteq Y$ be open. If $f^{-1}[U]$ is not open, then, $\alpha^{-1}[f^{-1}[U]]$ is not open as well since $X$ has the final topology with respect to $\alpha$, contradicting continuity of $f \circ \delta$. $f^{-1}[U]$ is open, hence $f$ is continuous. ☐

**Theorem 31** (Qualitative Main Theorem [Kreitz and Weihrauch, 1985])**.** *Let $X$ and $Y$ be second-countable $T_0$ spaces. Let $\delta_X$ and $\delta_Y$ be admissible representations of $X$ and $Y$, respectively. Let $f :\subseteq X \to Y$ be a partial function. Then $f$ is continuous if and only if $f$ admits a continuous $(\delta_X, \delta_Y)$-realizer $F :\subseteq \Sigma^\omega \to \Sigma^\omega$.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \delta_X}\uparrow & & \uparrow{\scriptstyle \delta_Y} \\ \Sigma^\omega & \xrightarrow{F} & \Sigma^\omega \end{array}$$

*Proof.* ($\Rightarrow$) Suppose that $f$ is continuous. Then $f \circ \delta_X$ is continuous. By reduction property of $\delta_Y$, we have $f \circ \delta_X \preceq_t \delta_Y$. In other words, there exists a continuous partial map $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $f \circ \delta_X = \delta_Y \circ F$.

($\Leftarrow$) Suppose that there exists a continuous partial map $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $f \circ \delta_X = \delta_Y \circ F$. Note that $\delta_Y \circ F$ is continuous, hence $f \circ \delta_X$ is continuous. Lemma 29 and Lemma 30 proves that $f$ is continuous. ☐

There are two ways to go about the proof of Theorem 31. One approach is by exploiting Corollary 23. By virtue of Corollary 23 and Theorem 28, if we succeed in proving the theorem only for standard representations $\delta_X$ and $\delta_Y$, then the theorem automatically holds for admissible representations $\delta_X$ and $\delta_Y$. The other approach is to prove that admissible representations fit into the topology of target space as final topology, and then prove Theorem 31 directly without help of standard representations. The two approaches are not so different, but we took the latter approach.

**Example 32** ([Kawamura et al., 2018])**.** Dyadic representation and signed binary representation are admissible

*Proof.* The proof is similar in spirit to the proof of Lemma 61. ☐

**Example 33.** Binary representation is not admissible

*Proof.* Try to reduce dyadic representation to binary representation and reach a contradiction. The proof is similar in spirit to the proof of Example 8. ☐

# Chapter 4. Quantitative Main Theorem

This chapter describes our main contribution, which we termed *Quantitative Main Theorem* of computable analysis. We need a few quantitative notions such as *modulus of continuity* of a partial function and *Kolmogorov entropy* [Kolmogorov and Tikhomirov, 1959] of a space before stating our quantitative main theorem.

## 4.1 Modulus of Continuity

A problem may or may not be computable. When it is computable, we are interested in its computational complexity: how difficult to compute it? In a similar manner, a function may or may not be continuous. When it is continuous, we are interested in its *modulus of continuity*: how discontinuous is it?

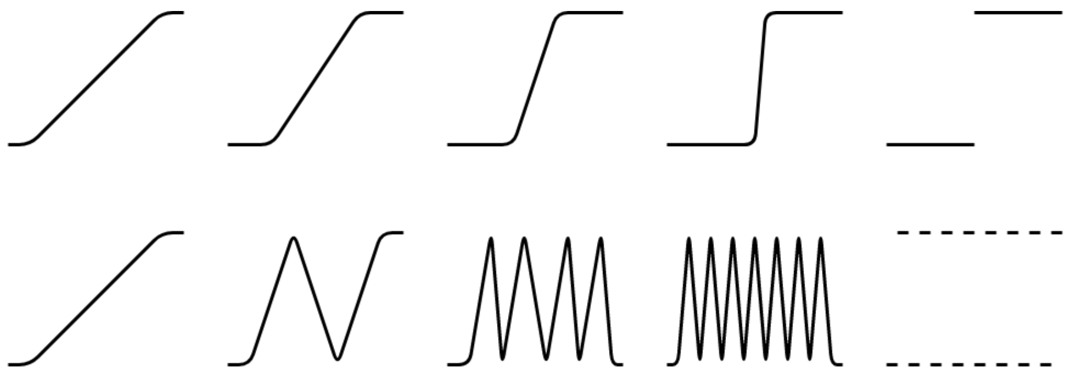


Figure 4.1: Two functions on the rightmost side are discontinuous. All others are continuous. However, these functions have some difference in *how much discontinuous* they are. Functions on the left are more continuous. Functions on the right are more discontinuous.

**Definition 34** (Modulus of Continuity [Steinberg, 2016])**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f :\subseteq X \to Y$ be a partial function. A map $\mu : \mathbb{N} \to \mathbb{N}$ forms a *modulus of continuity* of $f$ if

$$d_X(a, b) \leq 2^{-\mu(n)} \text{ implies } d_Y(f(a), f(b)) \leq 2^{-n}$$

for every $a, b \in \mathrm{dom}(f)$. Sometimes we simply say *modulus* instead of modulus of continuous.

Different from usual mathematical analysis, modulus of continuity is defined in terms of exponent of 2. This is for convenience of development of the theory.

Given $f$, its modulus of continuity is not unique; if $\mu(n) \leq \mu(n)'$ for all $n \in \mathbb{N}$ and $\mu$ forms a modulus of continuity of $f$, then so does $\mu'$. Generally, we are interested in whether or not $f$ has a modulus of continuity smaller than the given bound. Without loss of generality, we could always assume that modulus of continuity $\mu$ of $f$ is nondecreasing; for any modulus $\mu$, there exists a monotone increasing modulus $\mu'$ such that $\mu' \leq \mu$.

**Definition 35** (Minimum Modulus of Continuity)**.** $\mu$ is the *minimum modulus of continuity* of $f$ if

$$\mu \leq \nu$$

for every modulus $\nu$ of $f$.

If $f$ has a modulus, then it has the minimum modulus $\mu$. Explicitly, for each $n \in \mathbb{N}$,

$$\mu(n) = \min\{k \in \mathbb{N} \mid \forall a, b \in X : d_X(a, b) \leq 2^{-k} \Rightarrow d_Y(f(a), f(b)) \leq 2^{-n}\}.$$

A more curvy or fluctuating map $f$ corresponds to a larger or fast-growing minimum modulus of continuity and vice versa.

**Observation 36** ([Kawamura et al., 2018])**.** Let $X$ and $Y$ be metric spaces. Let $f :\subseteq X \to Y$ be a partial function.

1. Any $\mu$ forms a modulus of continuity for a constant function.
2. $f$ is uniformly continuous if and only if $f$ has a modulus of continuity.
3. Assume $\mathrm{dom}(f)$ to be bounded or connected. $f$ is Lipschitz continuous if and only if $f$ has modulus $\mu(n) = n + O(1)$.
4. Assume $\mathrm{dom}(f)$ to be bounded or connected. $f$ is Hölder continuous if and only if $f$ has modulus $\mu(n) = O(n)$.

The next lemma states that two moduli compose the reverse way the functions compose. The proof is by direct application of the definition of modulus of continuity.

**Lemma 37** (Modulus of Continuity of Composition)**.** *Let $f :\subseteq X \to Y$ have a modulus of continuity $\mu_f$. Let $g :\subseteq Y \to Z$ have a modulus of continuity $\mu_g$. Then their composition $g \circ f$ has a modulus of continuity $\mu_f \circ \mu_g$.*

*Proof.* Let $a, b \in X$. Suppose $d_X(a, b) \leq 2^{-\mu_f(\mu_g(n))}$. Then $d_Y(f(a), f(b)) \leq 2^{-\mu_g(n)}$. It follows that $d_Z(g(f(a)), g(f(b))) \leq 2^{-n}$. □

Note that Lemma 37 does not say that $\mu_f \circ \mu_g$ is the minimum modulus of continuity for $g \circ f$ even when $\mu_f$ and $\mu_g$ are the minimum moduli of continuity of $f$ and $g$, respectively. For example, let $f$ be a severely fluctuating map and $g$ be constant on $\mathrm{range}(f)$ while severely fluctuating on the other areas of $\mathrm{dom}(g)$. Then $g \circ f$ forms a constant map, whose the modulus of continuity $\mu_f \circ \mu_g$ is rather crude. More concretely, consider $f(x) = \sin(10^{10}x)$ and

$$g(y) = \begin{cases} 0 & (-1 \leq y \leq 1) \\ \sin(10^{10}y) & \text{otherwise} \end{cases}.$$

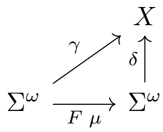
The following theorem states the relationship between time complexity and modulus of continuity. It provides us a justification for our main concerns being modulus of continuity, rather than computational complexity. The statement and the proof is the same in spirit as in Theorem 17, which is about qualitative properties: relationship between computability and continuity.

**Theorem 38** (Composition and Modulus of Continuity)**.** *Let $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ be a partial function. Let $t : \mathbb{N} \to \mathbb{N}$ be a total function. $F$ is computable with an oracle $\Sigma^* \to \Sigma^*$ in time $O(t(n))$ if and only if $F$ has modulus of continuity $O(t(n))$.*

*Proof.* The reasoning and construction of the oracle is the same as the proof of Theorem 17. Only a little more analysis on the length of finite strings is needed to complete the proof of time complexity and modulus of continuity. □

Definition 18 is about continuous reduction. The Definition 39 is a quantitative refinement of Definition 18.

**Definition 39** (Quantitative Reduction)**.** Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to X$ be two partial functions. Let $\mu : \mathbb{N} \to \mathbb{N}$ be a map. $\gamma$ *reduces* to $\delta$ *in* $\mu$, or, put another way, $\gamma$ $\mu$*-reduces* to $\delta$, if there exists a continuous partial function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\gamma = \delta \circ F$ and $\mu$ is a modulus of $F$.




When $\gamma$ $\mu$-reduces to $\delta$, it is denoted by $\gamma \preceq_\mu \delta$.

Care must be taken that continuous reduction $\preceq_t$ is not to be misinterpreted as quantitative $t$-reduction in an unbounded monotone increasing $t : \mathbb{N} \to \mathbb{N}$. The context will clarify the intended interpretation.

Suppose that $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to X$ have the minimum moduli $\mu_\gamma$ and $\mu_\delta$, respectively. Suppose that $\gamma \preceq_\mu \delta$ by a realizer $F$. By Lemma 37, $\mu \circ \mu_\delta$ forms a modulus of $\gamma$, hence we have $\mu_\gamma \leq \mu \circ \mu_\delta$. On the other hand, if $\mu \circ \mu_\delta \leq \mu_\gamma \circ (\mathrm{id} + c)$ holds for a constant $c \in \mathbb{N}$, then we could say that $F$ reduces $\gamma$ to $\delta$ in an optimal manner, in terms of its modulus of continuity.

**Definition 40** (Optimal Reduction)**.** Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to X$ be two uniformly continuous partial functions, having the minimum moduli $\mu_\gamma$ and $\mu_\delta$, respectively. $\gamma$ *optimally reduces to* $\delta$ if $\gamma \preceq_\mu \delta$ for some $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$\mu \circ \mu_\delta \leq \mu_\gamma \circ (\mathrm{id} + c)$$

for some $c \in \mathbb{N}$. Optimal reduction is symbolically denoted by $\gamma \preceq_{opt} \delta$.

$\gamma$ and $\delta$ are *optimally equivalent* if $\gamma \preceq_{opt} \delta$ and $\delta \preceq_{opt} \gamma$. Optimal equivalence is symbolically denoted by $\gamma \equiv_{opt} \delta$.

There is another equivalent formulation of Definition 40 that avoids saying *minimum* modulus.

**Observation 41** (Optimal Reduction Equivalent Formulation)**.** Let $\gamma :\subseteq \Sigma^\omega \to X$ and $\delta :\subseteq \Sigma^\omega \to X$ be two uniformly continuous partial functions. $\gamma \preceq_{opt} \delta$ holds if and only if for each modulus $\mu_\gamma$ of $\gamma$, there exists a modulus $\mu_\delta$ of $\delta$ and a map $\mu : \mathbb{N} \to \mathbb{N}$ such that $\gamma \preceq_\mu \delta$ and

$$\mu \circ \mu_\delta \leq \mu_\gamma \circ (\mathrm{id} + c)$$

for some $c \in \mathbb{N}$.

**Observation 42** (Optimal Reduction as a Relation)**.** Optimal reduction $\preceq_{opt}$ is reflexive and transitive. Optimal equivalence $\equiv_{opt}$ forms an equivalence relation.

Note that transitivity breaks down if optimal reduction $\preceq_{opt}$ were defined by a weaker condition

$$\mu \circ \mu_\delta \leq (\mathrm{id} + c) \circ \mu_\gamma \circ (\mathrm{id} + c)$$

in which $c$ appears in two different places.

One can measure modulus of continuity of a representation $\delta_X :\subseteq \Sigma^\omega \to X$ for a metric space $X$. Let $\mu : \mathbb{N} \to \mathbb{N}$ be a Kolmogorov entropy of $\delta_X$. For $p, q \in \Sigma^\omega$, $p$ and $q$ coincide for the first $k$ digits if and only if $d(p, q) \leq 2^{-k}$. That means, one has to know the name at least $\mu(n)$ digits to achieve precision of $2^{-n}$ in $X$. In a sense, $\mu$ somehow signifies "efficiency of encoding", or "density of information" of representation $\delta_X$. It would be very desirable to be able to identify each element of $X$ up to error $2^{-n}$ using possibly a small number $\mu(n)$ of digits.

Let us exclude the trivial case of bounded modulus.

**Observation 43** (Bounded Modulus of Continuity)**.** Let $\delta :\subseteq \Sigma^\omega \to X$ be a representation of a metric space $X$. If $\delta$ has a bounded modulus of continuity, then $X$ is finite.

There are a variety of spaces of our interest, which we want to encode by representations. Examples include: unit interval $[0, 1]$, unit square $[0, 1]^2$, unit cube $[0, 1]^3$, unit hypercube $[0, 1]^n$, Hilbert cube $\prod_{j\geq 0}[0, 2^{-j}]$, and space of 1-Lipschitz functions $Lip_1([0, 1], [0, 1])$ with domain and codomain being unit interval. These spaces seem to be different in their internal "structural complexity". Some of them seem to require more bits to achieve the same error bound than the others. $Lip_1([0, 1], [0, 1])$, for example, may not admit a representation having a modulus of continuity as small as that of the dyadic/signed binary representation of unit interval. Corollary 50 confirms this intuitive expectation formally.

## 4.2 Kolmogorov Entropy

Just like modulus of continuity is a quantitative measure about representations, Kolmogorov entropy [Kolmogorov and Tikhomirov, 1959] captures some quantitative properties of spaces.

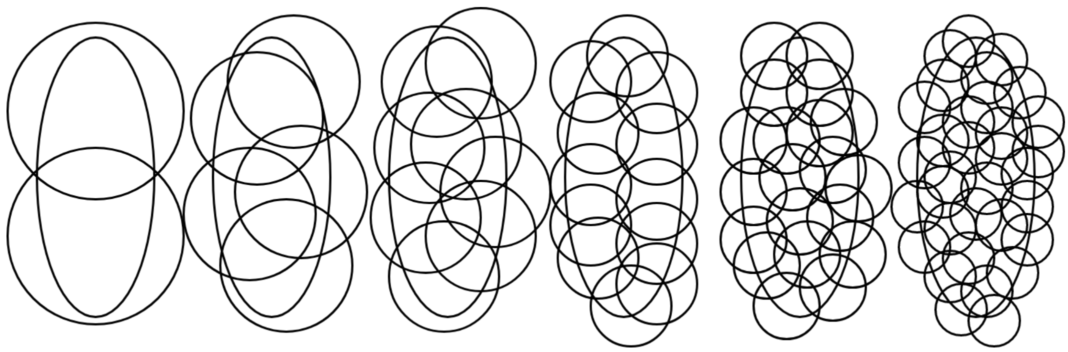


Figure 4.2: Kolomogorov Entropy

**Definition 44** (Kolmogorov Entropy [Kolmogorov and Tikhomirov, 1959])**.** Let $(X, d_X)$ be a metric space. *The Kolmogorov entropy* (or simply *entropy*) $\eta : \mathbb{N} \to \mathbb{N}$ is a total function, so that $X$ can be covered by $2^{\eta(n)}$ closed balls of radius $2^{-n}$, but not by $2^{\eta(n)-1}$ closed balls of radius $2^{-n}$. A *closed ball* $\overline{B}(x, \epsilon)$ refers to $\{y \mid d_X(x, y) \leq \epsilon\}$.

As in the case of Definition 34 of modulus of continuity, Kolmogorov entropy is defined in terms of exponent of 2. This is for convenience of development of the theory. Equivalently, Kolmogorov entropy $\eta(n)$ may be defined as $\lceil \log_2 M \rceil$, where $M$ is the minimum number of $2^{-n}$-closed balls needed to cover $X$.

A metric space admits at most one Kolmogorov entropy. Kolmogorov entropy is always monotone increasing. A metric space admits a Kolmogorov entropy if and only if the space is totally bounded.

**Observation 45.** Let $X$ be a metric space and $\eta$ be its entropy. $\eta$ is unbounded if and only if $X$ has infinitely many elements.

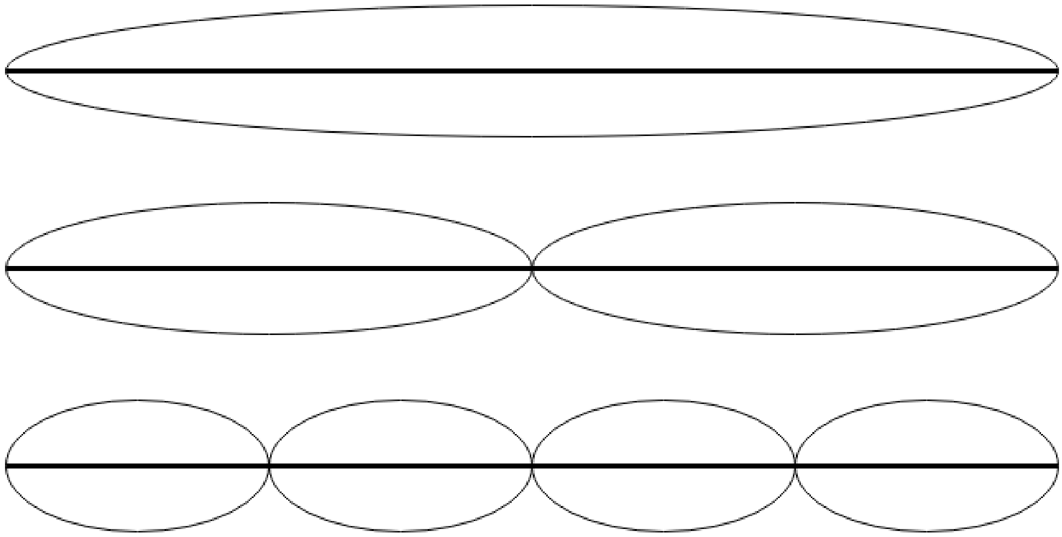


Figure 4.3: Kolomogorov Entropy of the Unit Interval; Half the radius, twice more balls.

**Example 46** (Spaces and Their Entropy [Kawamura et al., 2018]). One can sort out spaces according to their Kolmogorov entropy.

1. Unit hypercube $[0,1]^k$ has Kolmogorov entropy $\eta(n) = kn - 1$.

2. Hilbert cube $\prod_{j \geq 0}[0, 2^{-j}]$, with metric $(x, y) \mapsto \sup_{j \leq 0} |x_j - y_j|$, has Kolmogorov entropy $\eta(n) = \Theta(n^2)$.

3. The space $Lip_1([0,1],[0,1])$ of non-expansive functions has Kolmogorov entropy $\eta$ such that

$$2^{n-3} < \eta(n) < O(2^{n+2}).$$

4. The Cantor space $\Sigma^\omega$ has Kolmogorov entropy $\eta(n) = n$.

5. The compact subspace $\{0\} \cup \{\frac{1}{k} \mid k \in \mathbb{Z}^+\}$ in $\mathbb{R}$ has Kolmogorov entropy $\eta(n) = \lceil \log_2 n \rceil$.

**Example 47** (Change of Metric [Kawamura et al., 2018]). Kolmogorov entropy is heavily affected by change of metric to topologically equivalent one.

1. Let $X$ be a metric space with metric $d \leq 1$ and entropy $\eta$. Then

$$D(x, y) := \frac{1}{\log_2 2/d(x, y)}$$

constitutes a topologically equivalent metric yet inducing entropy $H(n) = \eta(2^n - 1)$.

2. Fix a nondecreasing unbounded map $\phi : \mathbb{N} \to \mathbb{N}$. Reconsider Cantor space $\Sigma^\omega$, now equipped with

$$d_\phi(p, q) := 2^{-\phi(\min\{j | p[i] \neq q[i]\})}.$$

$d_\phi$ constitutes a topologically equivalent metric to $d_{\text{id}}$ but with entropy $\eta = \underline{\phi^{-1}}$.

**Example 48** (Connected Spaces [Kawamura et al., 2018]). Every connected totally bounded metric space has entropy at least $\eta(n) = n + \Omega(1)$.

*Proof.* Let $X$ be a connected metric space having entropy $\eta$. Let $n \in \mathbb{N}$. Consider $x_1, \cdots x_{\eta(n)} \in X$ such that

$$X = \bigcup_j \overline{B}(x_j, 2^{-n}).$$

Consider the finite undirected graph $G = (V, E)$ with vertices $V = \{1, \cdots, \eta(n)\}$ and edges $E$ such that

$$\{i, j\} \in E \quad \Leftrightarrow \quad B(x_i, 2^{-n+1}) \cap B(x_j, 2^{-n+1}) \neq \emptyset.$$

This graph is connected: If $I, J \subseteq V$ were distinct connected components, then $\bigcup_{i \in I} B(x_i, 2^{-n+1})$ and $\bigcup_{i \notin I} B(x_i, 2^{-n+1})$ would form two disjoint open sets covering $X$. Therefore any two vertices are connected via $2^{\eta(n)} - 1$ edges; and for every edge $\{i, j\}$, it holds $d(x_i, x_j) < 2^{-n+2}$ by definition of the edge set $E$. Hence $x_i$ and $x_j$ have metric distance $d(x_i, x_j) < (2^{\eta(n)} - 1) \cdot 2^{-n+2}$; and for any $a, b \in X$ with $a \neq b$ have $d(a, b) \leq (2^{\eta(n)}) \cdot 2^{-n+2}$, requiring

$$d(a, b) \cdot 2^{n-2} \leq 2^{\eta(n)}.$$

□

There is a close connection between modulus of continuity and Kolmogorov entropy.

**Observation 49** (Modulus and Entropy [Steinberg, 2016])**.** Let $X$ and $Y$ be metric spaces having entropy $\eta_X$ and $\eta_Y$, respectively. Let $\mu$ be a modulus of continuity of a surjective partial function $f :\subseteq X \to Y$. Then we have

$$\eta_Y \leq \eta_X \circ \mu.$$

*Proof.* Fix $n \in \mathbb{N}$. It suffices to show that $Y$ can be covered by $\eta_X(\mu(n))$ closed balls of radius $2^{-n}$. Let $x_1, x_2, \cdots, x_m \in X$ be centers of closed balls of radius $2^{-\mu(n)}$ that cover $X$, satisfying $m \leq \eta_X(\mu(n))$. Then,

$$f(x_1), f(x_2), \cdots, f(x_m) \in Y$$

form centers of closed balls of radius $2^{-n}$ that cover $Y$. □

**Corollary 50** (Lower Bound of Moduli of Representations)**.** *Let $X$ be a metric space having entropy $\eta$. Let $\delta :\subseteq \Sigma^\omega \to X$ be a representation of $X$. Let $\mu$ be a modulus of continuity of $\delta$. Then, $\eta \leq \mu$.*

Informally, Corollary 50 may be interpreted that spaces with complicated internal structures cannot have efficient (or short) representations. The unit interval $[0, 1]$ has Kolmogorov entropy $\eta(n) = n - 1$. Various representations of $[0, 1]$ presented so far form examples of Corollary 50.

**Example 51** (Representations and Their Modulus [Kawamura et al., 2018])**.**

1. The binary representation of $[0, 1]$, as in Example 8, has a modulus of continuity $\mu(n) = n$.
2. The dyadic representation of $[0, 1]$, as in Definition 10, has a modulus of continuity $\mu(n) = \Theta(n^2)$.
3. The signed binary representation of $[0, 1]$, as in Definition 11, has a modulus of continuity $\mu(n) = \Theta(n)$.

## 4.3 Pseudoinverse

A total function $f : \mathbb{N} \to \mathbb{N}$ may or may not have the inverse. However, when $f$ is monotone increasing, $f$ admits something *like the inverse*. The notion of pseudoinverse is needed for succinct description of definitions than theorems that follow.

**Definition 52** (Pseudoinverse [Kawamura et al., 2018])**.** Let $\mu : \mathbb{N} \to \mathbb{N}$ be an unbounded monotone increasing map. The *pseudoinverse* $\mu^{\underline{-1}} : \mathbb{N} \to \mathbb{N}$ of $\mu$ is given by

$$\mu^{\underline{-1}}(n) = \min\{a \mid n \leq \mu(a)\}.$$

**Lemma 53** (Composition with Pseudoinverse [Kawamura et al., 2018])**.** *Let $\mu : \mathbb{N} \to \mathbb{N}$ be an unbounded monotone increasing map and $\mu^{\underline{-1}}$ be its inverse. Then,*

$$\mu^{\underline{-1}} \circ \mu \leq id_{\mathbb{N}} \leq \mu \circ \mu^{\underline{-1}}$$

.

## 4.4 Standard Representations

In this section we design *standard representations* of totally bounded metric spaces [Kawamura et al., 2016]. The basic idea is to generalization dyadic representation. Note that the unit interval, the target space of dyadic representation, is both totally bounded and metric.

Recall that *total boundedness* is the condition that the space can be convered by finitly many balls, however small the radius is; Definition 54 illustrates the need for total boundedness;

**Definition 54** (Radiuswise Enumeration)**.** Let $(X, d)$ be a totally bounded metric space having entropy $\eta$. A *Radiuswise enumeration* of $X$ is a sequence $(\xi_n :\subseteq \Sigma^{\eta(n+1)} \to X)_{n\in\mathbb{N}}$ of partial functions such that range$(\xi_n)$ form centers of closed balls having radius $2^{-n-1}$ covering X. More precisely, in mathematical expression,

$$X = \bigcup_{w\in \mathrm{dom}(\xi_n)} \overline{B}(\xi_n(w), 2^{-n-1}) \quad (n \in \mathbb{N}).$$

Note that in Definition 54, the ball radius for $\xi_n$ is $2^{-n-1}$, not $2^{-n}$. This deliberation is cleverly exploited in proofs of Lemma 59 and Lemma 61.

**Definition 55** (Rapidness)**.** A sequence $(x_n)_{n\in X}$ in a metric space $(X, d)$ is *rapid* if

$$d(x_n, x_m) \leq 2^{-n} + 2^{-m} \quad (n, m \in \mathbb{N}).$$

**Definition 56** (Rapid Convergence)**.** A sequence $(x_n)_{n\in X}$ in a metric space $(X, d)$ converges *rapidly* if $\lim_n x_n = x$ for some $x \in X$ such that

$$d(x, x_n) \leq 2^{-n} \quad (n \in \mathbb{N}).$$

**Lemma 57** (Rapid Convergence)**.** *A sequence $(x_n)_{n\in X}$ in a metric space $(X, d)$ converges rapidly if and only if it is rapid and converges.*

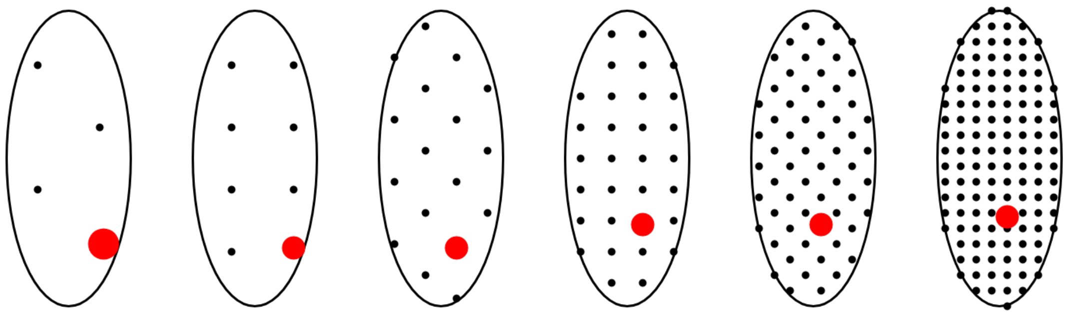


Figure 4.4: Standard Representation

**Definition 58** (Standard Representation of Totally Bounded Metric Space [Kawamura et al., 2016])**.** Let $(X, d)$ be a totally bounded metric space with unbounded Kolmogorov entropy $\eta$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a radiuswise enumeration of $X$. The *standard representation* $\delta :\subseteq \Sigma^\omega \to X$ with respect to a radiuswise enumeration $(\xi_n)$ is the partial map given by

$$w_0 w_1 \cdots w_n \cdots \mapsto \lim_n \xi_n(w_n)$$

where

$$\mathrm{dom}(\delta) = \{w_0 w_1 \cdots w_n \cdots \mid w_i \in \mathrm{dom}(\xi_i) \text{ and } (\xi_i(w_i))_{i\in\mathbb{N}} \text{ converges rapidly}\}.$$

For each $x \in X$, we can take a sequence $(x_n)$ in $X$ converging rapidly to $x$ such that $x_n \in \mathrm{range}(\xi_n)$, by Definition 54 of radiuswise enumeration. This ensures surjectivity of $\delta$.

A metric space is compact if and only if it is both complete and totally bounded. One alternative formulation is to assume the target space $X$ to be compact. Then can we better characterize the domain since

$$\begin{aligned}\mathrm{dom}(\delta) &= \{w_0 w_1 \cdots w_n \cdots \mid w_i \in \mathrm{dom}(\xi_i) \text{ and } (\xi_i(w_i))_{i\in\mathbb{N}} \text{ converges rapidly}\} \\ &= \{w_0 w_1 \cdots w_n \cdots \mid w_i \in \mathrm{dom}(\xi_i) \text{ and } (\xi_i(w_i))_{i\in\mathbb{N}} \text{ is rapid}\}.\end{aligned}$$

Given a sequence, its rapidness is much easier to check than rapid convergence. However, we will not concern ourselves with characterizations of domain. We will develop the theory without completeness.

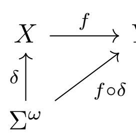
Standard representation of totally bounded metric spaces has many analogous properties as standard representation of $T_0$ second-countable spaces in Definition 24.

**Lemma 59** (Properties of Standard Representation of Totally Bounded Metric Spaces [Kawamura et al., 2018])**.** *Let $(X, d)$ be a metric space with entropy $\eta$. Let $\delta :\subseteq \Sigma^\omega \to X$ be the standard representation of $X$ with respect to a radiuswise enumeration $(\xi_n)$. Then,*

1. *$\delta$ has a modulus $\mu(n) = \sum_{i=0}^{n+1} \eta(i+1)$.*

2. *Let $n \in \mathbb{N}$. Let $x, x' \in X$ with $d(x, x') \le 2^{-n-2}$. Then there exists $p, p' \in \mathrm{dom}(\delta)$ with $d_{\Sigma^\omega}(p, p') \le 2^{-\mu(n)}$ such that $\delta(p) = x$ and $\delta(p') = x'$.*

3. *Let $(Y, e)$ be a totally bounded metric space. Let $f :\subseteq X \to Y$ be a partial map. If $f \circ \delta :\subseteq \Sigma^\omega \to Y$ has a modulus $\mu \circ \nu$, then $f$ has a modulus $\nu + 2$.*



*Proof.*

1. Let $p, p' \in \mathrm{dom}(\delta)$ with $d_{\Sigma^\omega}(p, p') \le 2^{-\mu(n)}$. Then $p$ and $p'$ have a common prefix $w_0 w_1 \cdots w_n w_{n+1}$. Then we have $d(\delta(p), \xi_{n+1}(w_{n+1})) \le 2^{-n-1}$ and $d(\delta(p'), \xi_{n+1}(w_{n+1})) \le 2^{-n-1}$. By triangular inequality, we have
$$d(\delta(p), \delta(p')) \le 2^{-n}.$$

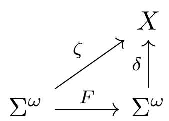
2. There exists $(w_i)_{i \in \mathbb{N}}$ such that
$$d(x, \xi_i(w_i)) \le 2^{-i-1} \quad (i \in \mathbb{N}).$$
Then we have
$$d(x', \xi_i(w_i)) \le d(x', x) + d(x, \xi_i(w_i)) \le 2^{-n-2} + 2^{-i-1} \le 2^{-i} \quad (i \le n+1).$$
Similarly, there exists $(w'_i)_{i > n+1}$ such that
$$d(x', \xi_i(w'_i)) \le 2^{-i} \quad (i > n+1).$$
Take
$$p = w_0 w_1 \cdots$$
and
$$p' = w_0 w_1 \cdots w_n w_{n+1} w'_{n+2} \cdots .$$

3. Suppose that $f \circ \delta$ has modulus $\mu \circ \nu$. Let $x, x' \in \mathrm{dom}(f) \subseteq X$ with $d(x, x') \le 2^{-\nu(n)-2}$. By item 2, we have $p, p' \in \mathrm{dom}(\delta)$ with $d_{\Sigma^\omega}(p, p') \le 2^{-\mu(\mu(n)')}$ such that $\delta(p) = x$ and $\delta(p') = x'$. Then,
$$e(f(x), f(x')) = e(f(\delta(p)), f(\delta(p'))) \le 2^{-n}.$$

□

**Lemma 60** (Subset Containment)**.** *Let $(X, d)$ be a metric space. Let $\epsilon > 0$. Let $A \subseteq X$ be such that*
$$\sup_{x,y \in A} d(x, y) \le \epsilon.$$
*Let $x_1, x_2, \cdots, x_m \in X$ be such that*
$$X = \bigcup_{i=1,\cdots,m} \overline{B}(x_i, \epsilon).$$
*Then for some $i$,*
$$A \subseteq \overline{B}(x_i, 2\epsilon).$$

**Lemma 61** (Properties of Standard Representation of Totally Bounded Metric Spaces [Kawamura et al., 2018])**.** *Let $(X, d)$ be a metric space with entropy $\eta$. Let $\delta :\subseteq \Sigma^\omega \to X$ be a standard representation of $X$. Let $\mu(n) = \sum_{i=0}^{n+1} \eta(i+1)$. Let $\zeta :\subseteq \Sigma^\omega \to X$ be a uniformly continuous partial map. Then*
$$\zeta \preceq_{opt} \delta.$$



*Proof.* Let $\mu_\zeta$ be the minimum modulus of $\zeta$. Let $(\xi_n :\subseteq \Sigma^{\eta(n+1)} \to X)_{n\in\mathbb{N}}$ be the radiuswise enumeration of $X$ from which $\delta$ is built. Fix $p \in \mathrm{dom}(\zeta)$. For each $n \in \mathbb{N}$, by Lemma 60, there exists $w_n \in \mathrm{dom}(\xi_n)$ so that

$$\zeta(p_{<\mu_\zeta(n+1)}\Sigma^\omega) \subseteq \overline{B}(\xi_n(w_n), 2^{-n}).$$

Define $F:\subseteq \Sigma^\omega \to \Sigma^\omega$ by

$$F(p) = w_0 w_1 w_2 \cdots .$$

The prefix $w_0 w_1 \cdots w_{n+1}$ of $F(p)$ of length $\mu(n) = \sum_{i=0}^{n+1} \eta(i+1)$ is completely determined by $p_{<\mu_\zeta(n+2)}$. $F$ has a modulus $m \mapsto \mu_\zeta(\mu^{\underline{-1}}(m) + 2)$. By Lemma 53 we have

$$\mu_\zeta(\mu^{\underline{-1}}(\mu(n)) + 2) \leq \mu_\zeta(n+2) \quad (n \in \mathbb{N}).$$

□

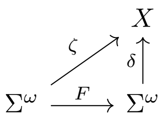
Compare Lemma 59 with Lemma 25; compare Lemma 61 with Lemma 26.

## 4.5 Quantitative Admissibility

This section is a quantitative analogue of Section 3.3.

**Definition 62** (Quantitative Admissibility [Kawamura et al., 2018])**.** Let $(X, d)$ be a totally bounded metric space. A representation $\delta$ of $X$ is *admissible* if

- $\delta$ is uniformly continuous and
- $\zeta \preceq_{opt} \delta$ for every uniformly continuous partial map $\zeta:\subseteq \Sigma^\omega \to X$




We use the same term *admissibility* for both the qualitative notion (Definition 27) and quantitative one (Definition 62). In case context cannot clarify the intended meaning, more wordy sentences are given.

Admissibility is just another way of characterizing the class of representations optimally equivalent to a standard representation.

**Theorem 63** (Equivalent Condition of Admissibility)**.** *Let $(X, d)$ be a totally bounded metric space. Let $\delta$ be a representation of $X$. Then $\delta$ is admissible if and only if $\delta$ is optimally equivalent to a standard representation of $X$.*

*Proof.* ($\Rightarrow$) Apply Definition 62, Lemma 59, and Lemma 61.

($\Leftarrow$) Apply the fact that composition of two uniformly continuous partial functions is uniformly continuous, Lemma 61, and Observation 42. □

We propose quantitative admissibility as a refinement of qualitative admissibility. Put the other way around, qualitative admissibility must be a generalization of quantitative admissibility. Observation 64 and 65 establishes this.

**Observation 64** (Refinement of Spaces)**.** Every totally bounded metric space is $T_0$ and second-countable.

**Observation 65** (Refinement of Admissibility)**.** Every quantitatively admissible representation is qualitatively admissible.

*Proof.* Let $(X, d)$ be a totally bounded metric space. Let $\sigma :\subseteq \Sigma^\omega \to X$ be a quantitative standard representation as in Definition 58, build from radiuswise enumeration $(\xi_n)$ of $X$. Let $\zeta :\subseteq \Sigma^\omega \to X$ be a continuous map. It suffices to show $\zeta \preceq_t \sigma$. The proof is a simpler version of the proof of Lemma 61.

Fix $p \in \mathrm{dom}(\zeta)$. For each $n \in \mathbb{N}$, by continuity of $\zeta$, there exists $m \in \mathbb{N}$ so that $\zeta(p_{<m}\Sigma^\omega)$ has diameter at most $2^{-n-1}$. Then, by Lemma 60, there exists $w_n \in \mathrm{dom}(\xi_n)$ so that

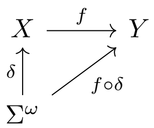
$$\zeta(p_{<m}\Sigma^\omega) \subseteq \overline{B}(\xi_n(w_n), 2^{-n}).$$

Note that $w_n$ depends only on a finite prefix of $p$. Define $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ by

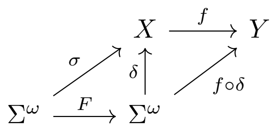
$$F(p) = w_0 w_1 w_2 \cdots .$$

Then $F$ is continuous with $\delta \circ F = \zeta$. □

**Lemma 66** (Recovering Modulus of Function)**.** *Let $X$ and $Y$ be totally bounded metric spaces. Let $f :\subseteq X \to Y$ be a partial function. Let $\delta :\subseteq \Sigma^\omega \to X$ be an admissible representation and $\mu_\delta$ be its minimum modulus. Consider an unbounded nondecreasing map $\mu_f : \mathbb{N} \to \mathbb{N}$. If $f \circ \delta$ has a modulus $\mu_\delta \circ \mu_f$, then $f$ has a modulus $\mu_f + c$ for some constant $c \in \mathbb{N}$.*




*Proof.*




Assume that $f \circ \delta$ has a modulus $\mu_\delta \circ \mu_f$. Consider a standard representation $\sigma :\subseteq \Sigma^\omega \to X$ and its (not necessarily minimum) modulus $\mu_\sigma(n) = \sum_{i=0}^{n+1} \eta(i+1)$, where $\eta$ is the entropy of $X$. By Definition 62, there exists a realizer $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\delta \circ F = \sigma$ admitting a modulus $\mu_F$ such that

$$\mu_F \circ \mu_\delta \leq \mu_\sigma \circ (\mathrm{id} + c)$$

for some constant $c \in \mathbb{N}$.

Note that $f \circ \sigma = f \circ \delta \circ F$ has a modulus $\mu_F \circ \mu_\delta \circ \mu_f$. By Lemma 53 we have

$$\mu_F \circ \mu_\delta \circ \mu_f \leq \mu_\sigma \circ \underline{\mu_\sigma}^{-1} \circ \mu_F \circ \mu_\delta \circ \mu_f.$$

The right side of the inequality above forms another modulus of $f \circ \sigma$. Lemma 59 tells us that

$$\underline{\mu_\sigma}^{-1} \circ \mu_F \circ \mu_\delta \circ \mu_f + 2$$

forms a modulus of $f$. Observe the following two inequalities:

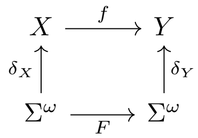
$$\begin{aligned} \underline{\mu_\sigma}^{-1} \circ \mu_F \circ \mu_\delta \circ \mu_f + 2 &\leq \underline{\mu_\sigma}^{-1} \circ \mu_\sigma \circ (\mathrm{id} + c) \circ \mu_f + 2 \\ &\leq \mu_f + c + 2. \end{aligned}$$

The first inequality holds by choice of $\mu_F$. The second holds by Lemma 53. $\mu_f + c + 2$ forms a modulus of $f$, as desired. □

In Lemma 66 and its proof, some moduli are minimum moduli while some are not minimum. Care should be taken regarding whether a modulus of continuity is the minimum one or not necessarily.

Finally, we state and prove the most important theorem in the thesis. Compare Theorem 67 with Theorem 31.

**Theorem 67** (Quantitative Main Theorem [Kawamura et al., 2018])**.** *Let $X$ and $Y$ be totally bounded metric spaces. Let $\delta_X$ and $\delta_Y$ be admissible representations of $X$ and $Y$, respectively. Let $\mu_X$ and $\mu_Y$ be the minimum moduli of $\delta_X$ and $\delta_Y$, respectively. Let $f :\subseteq X \to Y$ be a partial function.*




1. *If $f$ has a modulus $\mu_f$, then there exists a realizer $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ of $f$ admitting a modulus*

$$\mu_X \circ \mu_f \circ (id + c) \circ \mu_Y{}^{\underline{-1}}$$

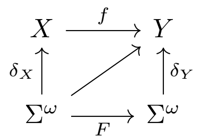
*for a constant $c \in \mathbb{N}$.*

2. *If $f$ has a realizer $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ having a modulus $\mu_F$, then $f$ admits a modulus*

$$(id + c) \circ \mu_X{}^{\underline{-1}} \circ \mu_F \circ \mu_Y$$

*for a constant $c \in \mathbb{N}$.*

*Proof.*




1. Assume that $f$ has a modulus $\mu_f$. Then the partial map $f \circ \delta_X{:}\subseteq \Sigma^\omega \to Y$ has a modulus $\mu_X \circ \mu_f$. Since $\delta_Y$ is admissible, there exists a realizer $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ having a modulus $\mu_F$ satisfying

$$\mu_F \circ \mu_Y \leq \mu_X \circ \mu_f \circ (\mathrm{id}_\mathbb{N} + c)$$

for some $c \in \mathbb{N}$. Observe the following inequalities:

$$\begin{aligned} \mu_F &\leq \mu_F \circ \mu_Y \circ \mu_Y{}^{\underline{-1}} \\ &\leq \mu_X \circ \mu_f \circ (\mathrm{id}_\mathbb{N} + c) \circ \mu_Y{}^{\underline{-1}}. \end{aligned}$$

The last term forms a modulus of $F$.

2. Assume that $f$ has a realizer $F{:}\subseteq \Sigma^\omega \to \Sigma^\omega$ having a modulus $\mu_F$. The map $f \circ \delta_X = \delta_Y \circ F$ has a modulus $\mu_F \circ \mu_Y$. By the following inequality

$$\mu_F \circ \mu_Y \leq \mu_X \circ \mu_X{}^{\underline{-1}} \circ \mu_F \circ \mu_Y$$

$\mu_X \circ \mu_X{}^{\underline{-1}} \circ \mu_F \circ \mu_Y$ forms a modulus of $f \circ \delta_X$. By Lemma 66, $\mu_X{}^{\underline{-1}} \circ \mu_F \circ \mu_Y + c$ forms a modulus of $f$, for some constant $c \in \mathbb{N}$.

□

$\mu_X$ and $\mu_Y$ are assumed to be the minimum moduli of continuity; this assumption plays a crucial role in the proof. Starting from $\mu_f$ and applying item 1 first and then 2 recovers

$$(\mathrm{id} + c) \circ \mu_f \circ (\mathrm{id} + c).$$

On the other hand, starting from $\mu_F$ and applying item 2 first and then 1 only yields

$$\mu_X \circ (\mathrm{id} + c) \circ \underline{\mu_X^{-1}} \circ \mu_F \circ \mu_Y \circ (\mathrm{id} + c) \circ \underline{\mu_Y^{-1}}.$$

## 4.6 Quantitative Product Representation

Let $X$ and $Y$ be totally bounded metric space, having admissible representations $\delta_X :\subseteq \Sigma^\omega \to X$ and $\delta_Y :\subseteq \Sigma^\omega \to Y$, respectively. The Cartesian product $X \times Y$ equipped with maximum metric is still totally bounded. One way to get an admissible representation for $X \times Y$ is to build a standard representation *from scratch.* There is, however, another way to construct a representation for $X \times Y$. We can combine already given representations $\delta_X$ and $\delta_Y$ into one. Usefulness of the constructions given in this section is illustrated in Chapter 5.

**Definition 68** (Code Splitter)**.** Let $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ be monotone increasing and unbounded. Define the total map *code splitter* $S_{\alpha,\beta} : \Sigma^\omega \to \Sigma^\omega \times \Sigma^\omega$ with respect to $\alpha$ and $\beta$ by

$$S(p) = (q, r)$$

where

$$q = p[0 \cdots \alpha(0)) \ \cdots p[\sum_{i=0}^{n}(\alpha(i) + \beta(i)) \cdots \sum_{i=0}^{n}(\alpha(i) + \beta(i)) + \alpha(n+1)) \ \cdots$$

and

$$r = p[\alpha(0) \cdots \alpha(0) + \beta(0)) \ \cdots p[\sum_{i=0}^{n}(\alpha(i) + \beta(i)) + \alpha(n+1) \cdots \sum_{i=0}^{n+1}(\alpha(i) + \beta(i))) \ \cdots .$$

**Observation 69** (Bijectivity of Code Splitters)**.** Every code splitter is bijective.

**Definition 70** (Quantitative Product Representation)**.** Let $X$ and $Y$ be totally bounded metric spaces. Let $\delta_X :\subseteq \Sigma^\omega \to$ and $\delta_Y :\subseteq \Sigma^\omega \to Y$ be admissible representations of $X$ and $Y$, having minimum moduli $\mu_{\delta_X}$ and $\mu_{\delta_Y}$, respectively. Define the *quantitative product representation* $\delta_X \times \delta_Y :\subseteq \Sigma^\omega \to X \times Y$ by

$$\delta_X \times \delta_Y(p) = (\delta_X(p_X), \delta_Y(p_Y))$$

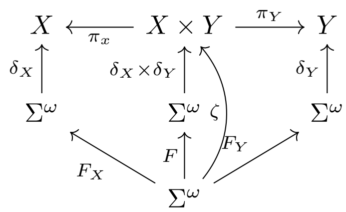
where

$$(p_X, p_Y) = S_{\mu_{\delta_X}, \mu_{\delta_Y}}(p).$$

$\mathrm{dom}(\delta_X \times \delta_Y)$ is defined canonically.

**Theorem 71** (Admissibility of Product Representation)**.** *The quantitative product representation, as in Definition 70, is quantitatively admissible.*

*Proof.*



Consider a uniformly continuous map $\zeta :\subseteq \Sigma^\omega \to X \times Y$ having minimum modulus $\mu_\zeta$. By quantitative admissibility of $\delta_X$, there exists $F_X :\subseteq \Sigma^\omega \to \Sigma^\omega$ with minimum modulus $\mu_{F_X}$ so that the diagram commutes and

$$\mu_{F_X} \circ \mu_{\delta_X} \leq \mu_\zeta \circ (\mathrm{id} + c_X)$$

for some constant $c_X$. Similarly, there exists $F_Y :\subseteq \Sigma^\omega \to \Sigma^\omega$ with minimum modulus $\mu_{F_Y}$ such that

$$\mu_{F_Y} \circ \mu_{\delta_Y} \leq \mu_\zeta \circ (\mathrm{id} + c_Y)$$

for some constant $c_Y$. Combining the two inequalities above, we have

$$\max(\mu_{F_X} \circ \mu_{\delta_X}, \mu_{F_Y} \circ \mu_{\delta_Y}) \leq \mu_\zeta \circ (\mathrm{id} + \max(c_X, c_Y))$$

where maximum operation on the left is pointwise. Now our goal is to construct $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ with minimum modulus $\mu_F$ so that the diagram commutes and

$$\mu_F \circ \mu_{\delta_X \times \delta_Y} \leq \max(\mu_{F_X} \circ \mu_{\delta_X}, \mu_{F_Y} \circ \mu_{\delta_Y}).$$

So let us define $F$ by

$$F(p) = S^{-1}_{\mu_{\delta_X}, \mu_{\delta_Y}}(F_X(p), F_Y(p)).$$

Observe that

$$\begin{aligned}\mu_F(\mu_{\delta_X \times \delta_Y}(n)) &= \mu_F(\mu_{\delta_X}(n) + \mu_{\delta_Y}(n)) \\ &= \max(\mu_{F_X}(\mu_{\delta_X}(n)), \mu_{F_Y}(\mu_{\delta_Y}(n))).\end{aligned}$$

□

# Chapter 5. Categories of Represented Spaces

One common definition of a represented space in the literature is a pair $(X, \delta)$ of a space $X$ and a representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ [Pauly and Brecht, 2015] [Pauly, 2015]. Baire space $\mathbb{N}^{\mathbb{N}}$ is more commonly used as a ground set than Cantor space $\Sigma^\omega$. $X$ is assumed to have the final topology with respect to $\delta$. Arrows of a category of represented spaces are functions that have computable or continuous realizers. Realizers themselves are not part of an arrow.

This chapter formulates represented spaces as categories. One main difference of our approach from previous research is that arrows of a category are pairs $(f, F)$ of a functions and its realizer.

Consider two represented spaces $(X, \delta_X)$ and $(Y, \delta_Y)$, and a represented function space $(Y^X, \delta_{Y^X})$. Assume that the ground set for representations is the Cantor space $\Sigma^\omega$. Consider a representation $\gamma :\subseteq \Sigma^\omega \to X \times Y^X$, given by

$$\gamma(p) = (\delta_X(p_0p_3p_6\cdots), \delta_{Y^X}(p_1p_4p_7\cdots)),$$

where

$$\begin{aligned}\mathrm{dom}(\gamma) = \{p \in \Sigma^\omega \mid & p_0p_3p_6\cdots \in \mathrm{dom}(\delta_X),\\ & p_1p_4p_7\cdots \in \mathrm{dom}(\delta_{Y^X}),\\ & p_2p_5p_8\cdots \in \mathrm{dom}(\delta_Y), \text{ and}\\ & \delta_{Y^X}(p_1p_4p_7\cdots)(\delta_X(p_0p_3p_6\cdots)) = \delta_Y(p_2p_5p_8\cdots)\}.\end{aligned}$$

Have a careful look that the domain. Simply put, for $p \in \mathrm{dom}(\gamma)$,

- $p_0p_3p_6\cdots$ corresponds to $x \in X$;
- $p_1p_4p_7\cdots$ corresponds to $f \in Y^X$;
- $p_2p_5p_8\cdots$ corresponds to $f(x) \in Y$.

The information contained 1/3 of the bit sequence is redundant as an encoding for $X \times Y^X$. However, consider the evaluation map $\mathrm{eval} : X \times Y^X \to Y$. If we were to use $\gamma$ as the representation for $X \times Y^X$, then the evaluation map would have a trivial realizer. This is a weird observation.

We want to illegalize such a representation $\gamma$. The representation for Cartesian product $X \times Y$ should be constructed from representations of $X$ and $Y$. One way out is to define arrows of a category of represented spaces by pairs $(f, F)$ of a function $f$ and its realizer $F$, as will be shown in this chapter.

## 5.1 Category of Represented Topological Spaces

**Definition 72** (Category of Represented Topological Spaces)**.** The *category of represented topological spaces*, denoted by **RTop**, is described below.

**Objects** pairs $(X, \delta)$ of a second-countable $T_0$ topological space $X$ and an admissible representation $\delta :\subseteq \Sigma^\omega \to X$

**Arrows** pairs $(f, F) : (X, \delta_X) \to (Y, \delta_Y)$ of a total continuous function $f : X \to Y$ and a continuous realizer $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ satisfying $\mathrm{dom}(F) = \mathrm{dom}(\delta_X)$ and $\mathrm{range}(F) \subseteq \mathrm{dom}(\delta_Y)$

**Composition** $(g, G) \circ (f, F) = (g \circ f, G \circ F)$

**RTop** forms a category. It is closed under composition. The identity arrow of object $(X, \delta)$ is $(\mathrm{id}_X, \mathrm{id}_{\mathrm{dom}(\delta)})$. Note that the restriction of $\mathrm{range}(F) \subseteq \mathrm{dom}(\delta_Y)$ is redundant since $F$ is a realizer and must satisfy $f \circ \delta_X = \delta_Y \circ F$.

In previous development outside the framework of category theory, $f$ need not be total and $\mathrm{dom}(F) = \mathrm{dom}(\delta_X)$ need not hold. We need these additional restrictions to prove categorical properties of **RTop**.

For any two objects $(X, \delta_X)$, $(Y, \delta_Y)$ and a continuous total map $f : X \to Y$, Theorem 31 ensures that there is a realizer $F$ such that $(f, F) : (X, \delta_X) \to (Y, \delta_Y)$ forms an arrow.

**Observation 73** (Isomorphic Objects)**.** If $(X, \delta_X)$ and $(Y, \delta_Y)$ are isomorphic. then $X$ and $Y$ are homeomorphic; $\mathrm{dom}(\delta_X)$ and $\mathrm{dom}(\delta_Y)$ are homeomorphic. The other way around is not true.

**Observation 74** (Final Objects)**. RTop** has final objects. An object $(X, \delta)$ is final if $|X| = |\mathrm{dom}(\delta)| = 1$. Note that $\delta$ is trivially admissible in that case.

The product space of finitely or countably many second-countable $T_0$ spaces is again second-countable and $T_0$, under product topology. These facts suggest that **RTop** may have categorical products. This is true indeed as shown by Theorem 75 and 76. The construction of *canonical* representations is from [Weihrauch, 2000].

**Theorem 75** (Binary Product)**.** ***RTop*** *has binary products.*

*Proof.* Let $(X, \delta_X)$ and $(Y, \delta_Y)$ be two objects of **RTop**. Define a representation $\delta_{X\times Y} :\subseteq \Sigma^\omega \to X \times Y$ constructed from $\delta_X$ and $\delta_Y$ by

$$\delta_{X\times Y}(p) = (\delta_X(p_0p_2p_4\dots), \delta_Y(p_1p_3p_5\cdots)),$$

where

$$\mathrm{dom}(\delta_{X\times Y}) = \{p \in \Sigma^\omega \mid p_0p_2p_4\cdots \in \mathrm{dom}(\delta_X) \text{ and } p_1p_3p_5\cdots \in \mathrm{dom}(\delta_Y)\}.$$

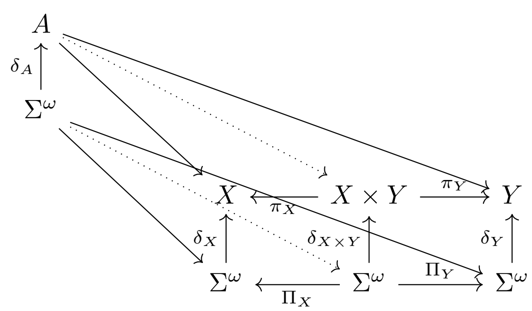
Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be set projections. Define a realizer $\Pi_X : \mathrm{dom}(\delta_{X\times Y}) \to \Sigma^\omega$ of $\pi_X$ by

$$\Pi_X(p) = p_0p_2p_4\cdots.$$

Similarly, define a realizer $\Pi_Y : \mathrm{dom}(\delta_{X\times Y}) \to \Sigma^\omega$ of $\pi_Y$ by

$$\Pi_Y(p) = p_1p_3p_5\cdots.$$

Then $(X \times Y, \delta_{X\times Y})$ with projections $(\pi_X, \Pi_X)$ and $(\pi_Y, \Pi_Y)$ forms a categorical binary product. Consider the commutative diagram below.




□

**Theorem 76** (Countable Product)**.** ***RTop** has countable products.*

*Proof.* Let $(X_i, \delta_i)$ be an object of **RTop** for every $i \in \mathbb{N}$. Denote the set product $\prod_i X_i$ by $X$. Define a representation $\delta :\subseteq \Sigma^\omega \to X$ constructed from $\delta_i$'s by

$$\delta(p) = (\delta_i(p_{\langle i,0\rangle} p_{\langle i,1\rangle} p_{\langle i,2\rangle} \cdots))_{i\in\mathbb{N}},$$

where

$$\mathrm{dom}(\delta) = \{p \in \Sigma^\omega \mid p_{\langle i,0\rangle} p_{\langle i,1\rangle} p_{\langle i,2\rangle} \cdots \in \mathrm{dom}(\delta_i) \text{ for every } i \in \mathbb{N}\}.$$

Everything else is almost the same as proof of Theorem 75. □

Second-countability and $T_0$ are closed under disjoint sums, finite or countable. **RTop** has binary and countable sums as well. The construction of canonical representations is from [Weihrauch, 2000].

**Theorem 77** (Binary Sum)**.** ***RTop** has binary sums.*

*Proof.* This proof is analogous to the proof of Theorem 75. Let $(X, \delta_X)$ and $(Y, \delta_Y)$ be two objects of **RTop**. Define a representation $\delta_{X+Y} :\subseteq \Sigma^\omega \to X + Y$ constructed from $\delta_X$ and $\delta_Y$ by

$$\delta_{X+Y}(p) = \begin{cases} \delta_X(p_1 p_2 p_3 \cdots) & p_0 = \mathtt{0} \\ \delta_Y(p_1 p_2 p_3 \cdots) & p_0 = \mathtt{1} \end{cases}$$

where

$$\mathrm{dom}(\delta_{X+Y}) = \{\mathtt{0}p \in \Sigma^\omega \mid p_1 p_2 p_3 \cdots \in \mathrm{dom}(\delta_X)\} \cup \{\mathtt{1}p \in \Sigma^\omega \mid p_1 p_2 p_3 \cdots \in \mathrm{dom}(\delta_Y)\}.$$

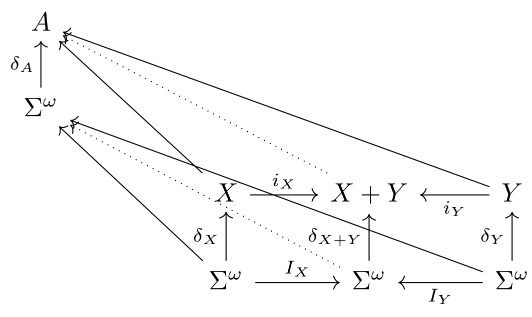
Let $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$ be set injections. Define a realizer $I_X : \mathrm{dom}(\delta_X) \to \Sigma^\omega$ of $i_X$ by

$$I_X(p) = \mathtt{0}p.$$

Similarly, define a realizer $I_Y : \mathrm{dom}(\delta_Y) \to \Sigma^\omega$ of $i_Y$ by

$$I_Y(p) = \mathtt{1}p.$$

Then $(X + Y, \delta_{X+Y})$ with injections $(i_X, I_X)$ and $(i_Y, I_Y)$ forms a categorical binary product. Consider the commutative diagram below.




□

**Theorem 78** (Countable Sum)**.** ***RTop** has countable sums*

*Proof.* The proof is analogous to the proof of Theorem 76. Let $(X_i, \delta_i)$ be an object of **RTop** for every $i \in \mathbb{N}$. Denote the set disjoint union $\sum_i X_i$ by $X$. Define a representation $\delta :\subseteq \Sigma^\omega \to X$ constructed from $\delta_i$'s by

$$\delta(p) = \delta_n(p_{n+1} p_{n+2} p_{n+3} \cdots),$$

where $n \in \mathbb{N}$ is the largest number such that $p_{<n} = \mathtt{1}^n$ and $\mathrm{dom}(\delta)$ is defined canonically

Everything else is almost the same as proof of Theorem 77. □

## 5.2 Category of Represented Metric Spaces

This section is an analogue of Section 5.1. We take totally bounded metric spaces as objects; pairs of a 1-Lipschitz map and its uniformly continuous realizer as arrows.

**Definition 79** (Category of Represented Metric Spaces)**.** The *category of represented metric spaces*, denoted by **RMet**, is described below.

**Objects** pairs $(X, \delta)$ of a totally bounded topological space $X$ and an admissible representation $\delta :\subseteq \Sigma^\omega \to X$

**Arrows** pairs $(f, F) : (X, \delta_X) \to (Y, \delta_Y)$ of a total 1-Lipschitz function $f : X \to Y$ and a uniformly continuous realizer $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ satisfying $\mathrm{dom}(F) = \mathrm{dom}(\delta_X)$ and $\mathrm{range}(F) \subseteq \mathrm{dom}(\delta_Y)$

**Composition** $(g, G) \circ (f, F) = (g \circ f, G \circ F)$

**RTop** forms a category. It is closed under composition. The identity arrow of object $(X, \delta)$ is $(\mathrm{id}_X, \mathrm{id}_{\mathrm{dom}(\delta)})$. Note that, as in the case of Definition 72, the restriction of $\mathrm{range}(F) \subseteq \mathrm{dom}(\delta_Y)$ is redundant since $F$ is a realizer and must satisfy $f \circ \delta_X = \delta_Y \circ F$.

$f$ is restricted to 1-Lipschitz maps to ensure that categorically isomorphic objects have isometric metric spaces, as in Observation 80.

For any two objects $(X, \delta_X)$, $(Y, \delta_Y)$ and a 1-Lipschitz total map $f : X \to Y$, Theorem 67 ensures that there is a realizer $F$ such that $(f, F) : (X, \delta_X) \to (Y, \delta_Y)$ forms an arrow.

**Observation 80** (Isomorphic Objects)**.** If $(X, \delta_X)$ and $(Y, \delta_Y)$ are isomorphic. then $X$ and $Y$ are isometric; $\mathrm{dom}(\delta_X)$ and $\mathrm{dom}(\delta_Y)$ are homeomorphic. The other way around is not true.

**Observation 81** (Final Objects)**. RMet** has final objects. An object $(X, \delta)$ is final if $|X| = |\mathrm{dom}(\delta)| = 1$. Note that $\delta$ is trivially admissible in that case.

Consider two totally bounded metric spaces $X$ and $Y$. The Cartesian product $X \times Y$ equipped with maximum metric is again totally bounded. This $X \times Y$, with an appropriate representation constructed from representations of $X$ and $Y$, forms a categorical binary product. Contrary to Section 5.1, there is no appropriate countable product for totally bounded metric spaces.

**Theorem 82** (Binary Product)**.** ***RMet** has binary products.*

*Proof.* Similar to the proof of Theorem 75. Quantitative product representation, as in Definition 70, is used instead of the naive product representation. □

The category **RMet** has all finite products. Could we aim for Cartesian closure of **RMet**? We need exponentials. The Observation 83 suggests at least a possibility. Its proof is a simpler version of the proof of Arzelà-Ascoli theorem [Munkres, 2000].

**Observation 83** (Closure of Total Boundedness by Exponential)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be totally bounded metric spaces. Consider an arbitrary constant $c \in \mathbb{R}^+$. $Lip_c(X, Y)$, equipped with supremum metric, is totally bounded.

*Proof.* A metric space is totally bounded if and only if every infinite subspace has arbitrarily close pairs of points. Consider an infinite subset $\mathcal{F} \subseteq Lip_c(X, Y)$. It suffices to prove that there exists arbitrary close $f, g \in \mathcal{F}$.

Let $\epsilon > 0$. Let $y_1, y_2, \cdots, y_m \in Y$ be such that.

$$Y = \bigcup_{1 \le i \le m} \overline{B}(y_i, \epsilon).$$

Similarly, let $x_1, x_2, \cdots, x_m \in X$ be such that

$$X = \bigcup_{1 \le i \le m} \overline{B}(x_i, \epsilon).$$

For every arbitrary $x \in X$, there exists $y_i$ such that

$$\mathcal{F}_x = \{f \in \mathcal{F} \mid f(x) \in \overline{B}(y_i, \epsilon)\}$$

is infinite. Apply this procedure iteratively to $\mathcal{F}$ for $x_1, x_2, \cdots x_m$, one by one. We end up with an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ such that for each $f, g \in \mathcal{G}$ we have, for all $i = 1, \cdots, m$,

$$d_Y(f(x_i), g(x_i)) \le 2\epsilon$$

Let $f, g \in \mathcal{G}$. Let $x \in X$ be arbitrary. There exists $x_i$ with $d(x, x_i) \le \epsilon$. We further have

$$\begin{aligned} d_Y(f(x), f(x_i)) &\le c \cdot d_X(x, x_i) \le c\epsilon; \\ d_Y(f(x_i), g(x_i)) &\le 2\epsilon; \\ d_Y(g(x_i), g(x)) &\le c \cdot d_X(x, x_i) \le c\epsilon. \end{aligned}$$

Triangular inequality yields

$$d_Y(f(x), g(x)) \le 2\epsilon(1 + c).$$

□

# Chapter 6. Limitations and Future Work

We proposed *quantitative admissibility* (Definition 62) as a criterion for representations of totally bounded metric spaces. One limitation of its definition is that it does not impose any restriction on the modulus of continuity For example, consider an admissible representation $\delta :\subseteq \Sigma^\omega \to X$ of a totally bounded metric space $X$. Consider a possibly very fast-growing map $\varphi : \mathbb{N} \to \mathbb{N}$. We can define another representation $\gamma :\subseteq \Sigma^\omega \to X$ by

$$\gamma(p) = \delta(p_{\varphi(0)} p_{\varphi(1)} p_{\varphi(2)} \cdots).$$

The representation $\gamma$ ignores possibly too many bits. The ignored bits contain no information at all. The modulus of $\gamma$, which may be considered *efficiency* of a representation, is possibly extremely fast-growing. Yet, $\gamma$ still fulfills the criteria of admissibility. $\gamma$ is certainly not a good representation. There must be a way to exclude $\gamma$ from the class of admissible representations.

The category **RMet** has as arrows 1-Lipschitz functions with their realizers. Perhaps it does not include a very large class of functions of general interest. The possibility of Cartesian closure of **RMet** is open, as suggested by Observation 83. Representations of functions spaces may be designed out of already existing representations to form an exponential object, as in the case of categorical product. Other category-theoretic properties should be investigated further.

The current work is only about continuity, rather than computability. For concreteness of the theory, computability of realizers should be investigated. Imposing computability conditions on radiuswise enumerations (Definition 54) seems a good starting point.

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# Acknowledgment

This work was presented at WAAC 2018, which stands for *japan-korea joint Workshop on Algorithms And Computation*. Do you see? In the acronym *WAAC*, *W* stands for workshop, *A* stands for algorithms, *A* stands for and, and *C* stands for computation. Even the conjunction *and* occupies a character, while there is no place for *Japan* and *Korea*, the names of host nations. What is behind the scene—which I believe is true even though I do not have any evidence—is that they had no choice since the contemporary writing system of English (nor Japanese, nor Korean, for that matter) does not have any method to write two characters simultaneously. One party would have been unhappy if it were *JKWAAC*; the other party would have been unhappy if it were *KJWAAC*. They made a compromise and decided to alternate between two slightly different full names *Japan-Korea Joint Workshop on Algorithms and Computation* and *Korea-Japan Joint Workshop on Algorithms and Computation*, while keeping the acronym *WAAC* for consistency.

Figure 6.1: Official Logo of 2002 FIFA World Cup

One alternative is to conform to the alphabetical order. However, as it had been shown in the case of 2002 FIFA World Cup, the problem is not so simple. FIFA (Fédération Internationale de Football Association), the organizing body of World Cup, is originated from France in 1904. Korea is *Corée* in French; Japan is *Japon* in French. *C* comes before *J* in French alphabet, hence the order *Korea Japan* in the official logo. Not to mention, the Japanese people were very unhappy.

*Coordinated Universal Time* is abbreviated as *UTC*, not *CUT*. Originally, French speakers proposed *TUC*, for *temps universel coordonné*; English speakers proposed *CUT*, for *Coordinated Universal Time*. They compromised to *UTC*, an acronym of nothing.

Which writing system to use to decide the order? Certainly English system is not the only answer, as shown by two examples above.

In North Korea, they say *North-South* relationship. In South Korea, they say *South-North* relationship. When it comes to the relationship between North Korea and the United States, South Korean people mostly say *North Korea-United States* relationship. Some people argue that this practice is wrong, considering the economic and military support by the United States as a guardian of liberal democracy, in contrast to the infuriating mischief of North Korea, such as dictatorship, brinkmanship, and nuclear weapons. This opinion is based on the assumption that "the better should literally precede the worse".

What I find a bit puzzling is that the field of acknowledgement writing has an unwritten law: the names of family, relatives, friends, significant other, cats, and dogs appear the last, in contrast to early appearance of names of colleagues and editors, who made concrete contributions to the thesis or the book. I have observed no exception. Is it because they are the least important? In my case, my family has no working knowledge of mathematics or computer science (no offense). They are unlikely to make a meaningful scientific contribution (no offense). Is this the reason why they are placed the last? However, I do not share my family with other scientists. Maybe some scientists have a member of their family who knows science.

Do I have a great misunderstanding? It might be the case that they are placed the last, not because they are the least important, but because they are the most important. Heroes always come at the last moment. They are the source of emotional fulfillment, which is way more important in life than science. One cannot do a good research while being emotionally unstable; stability is a matter of life and death.

They are placed last but not least; they are the best. Although they usually forbid in courses of English writing the use of cliches such as this one. Even in the field of journal publication, at times the most important contributor occupies the last place. He may be the last, but certainly not the least.

Order of names is of utmost importance to all of us. We cannot simply settle to an arbitrary order. We are social animals. We are hardwired to pay attention to the matters of fame and reputation. Our brains have evolved that way. We are always concerned who gets what and who deserves what. This is the eternal theme of human narrative.

Organizers of WAAC decided to alternate the order annually. In principle, I *could* do the same thing. I am planning to distribute this thesis to my acquaintances; for each of them, I *could* prepare a different version, just the right thing for him, in which his name comes the first (or the last). Everybody will be happy to read the acknowledgement and realize that his name comes the first (or the last), *except I myself.* While you being just happy, I have to do the all the work of preparing different versions and pay for additional printing fee.

I made a selfish decision. I will prepare only a single version. No name of my acquaintances will appear in this acknowledgement. Organizers of WAAC did the same trick; nobody could blame me as long as they can't blame organizers of WAAC. I know it is the logically fallacy of *tu quoque*, appeal to hypocrisy, to try to justify my wrongdoings by blaming others. I am just too much of a coward to choose the order. I am very very sorry about this pathetic decision, to all those who are expecting their names in this acknowledgement. I think I have given more than enough justification for my cowardice throughout this rather long acknowledgement, which I wonder anybody would ever read until the end.

Yes, I am such a chicken; a chicken is what I am. (Unrelated, I was born in a year of chicken. Certainly many people born in the same year (mod 12) are not chickens. A human cannot be a chicken anyway. My brother loves fried chickens. He was born in a year of dog.) Please, please forgive me. What I wanted is just not to make you angry by failing to put your name in the first place (or the last place). I know good intentions do not necessarily lead to good results, but at least I had good intentions.

Thank you all for your patience and understanding. I sincerely appreciate. My excuse was way longer than necessary. Now let me express my gratitude from now on.

Without all your (some of them dead) support, whether scientific or emotional, whether sincere or not so sincere—not-so-good intentions may as well lead to good results, as in the case of Richard Feynman, a Nobel-prize-winning scientist, who worked for his own fun of it, not for benefits of humanity; another example is the selfishness of players participating in the market, with each player working toward benefits of his own, eventually leading to growth of the whole economy by marvelous power of the invisible hand, though being criticized by socialist economists—this work had never been possible. Put another way, your wrongdoings led to production of such a pathetic work, barely counted as a scientific contribution but rather regarded as a contribution to deluge of information.

Some authors—after finishing such a great piece of work—argue that if there is any virtue in their

work, it is by the support of other people, while any vice is their own. This is a logical fallacy I think. You can't have your cake and eat it too.

I am now in a dilemma. I can attribute everything, virtue and vice, either to myself or to you (some of them dead). The former choice, that I did all the work on my own without your (some of them dead) support, can hardly constitute an acknowledgement. The latter choice, that all the work is by your (some of them dead) support, can be interpreted in two ways:

- Your (some dead) support produced a great work.
- Your (some dead) support produced a pathetic work.

One professor of theoretical computer science—my adviser—told me, "my master thesis, it is an embarrassment now". Is it feasible for mine to be a great piece of work while his being an embarrassment? As I told you, this work would barely be counted as a scientific contribution but rather as a contribution to deluge of information. Nobody would read this thesis except the referees. The first interpretation is ridiculous.

The effect of attribution depends strongly on the quality of the work. If you succeeded in creating a masterpiece, then you can be assured that your attribution hardly signifies an insult. In my case, I should be warned that the same conduct may result in inappropriate behavior.

I need to tell you one more thing. A tiny portion of my master thesis is original to myself. I was hardly able to discover something that wasn't there before. Ergo, the act of my undermining this work may undermine the things that were already there before. It is not my intention. This consideration complicates the situation even more that is way too much more complicated even now than I could ever handle on my own. Complexity is something that I have never been able to deal with.

My plan was to express my gratitude. It is by my lack of social skill that I am struggling to come up with an appropriate and effective method. Once again, can I just rely on good intentions that may or may not lead to good results? I know that this is another logical fallacy of appeal to emotion, or *argumentum ad passiones*. I Looked up Wikipedia for Latin. I don't know Latin.

Thank you all (including but not limited to the dead). I am not sure whether I should say it is by your (including but not limited to the dead) support that I made it until now as illustrated by the dilemma above. Your (including but not limited to the dead) jokes—whether intended or not—made me laugh. Your (including but not limited to the dead) encouragement made me finish this work that I have never imagined possible—while actually not having finished the work at the moment of writing this sentence. Thank you (including but not limited to the dead) for your (including but not limited to the dead) care.

본 연구는 WAAC 2018에서 발표되었습니다. $WAAC$라는 약어는 japan-korea joint Workshop on Algorithms And Computation(알고리즘과 계산에 관한 한일 공동 워크숍)을 뜻합니다. 잘 보세요. $WAAC$에서, $W$는 워크숍, $A$는 알고리즘, $A$는 and, $C$는 계산을 뜻합니다. 접속사에 불과한 $and$도 한 문자를 차지하는데, 개최국인 한국과 일본은 아무 자리도 차지하지 못하고 있습니다. 아마도 현대의 영어 표기체계에는 (혹은 한국어든 일본어든 뭐든) 문자 두 개를 동시에 쓸 수 있는 방법이 없어서 그런 것 아닐까요? 만약 $JKWAAC$였다면 한쪽이 매우 불행해했을 것이고, $KJWAAC$였다면 다른 한쪽이 매우 불행해했을 것입니다. 비록 아무런 증거도 없지만, 저는 물밑에서 다음과 같은 일이 일어났다고 확신합니다. 그들은

타협했습니다. 두 학술대회명, Japan-Korea Joint Workshop on Algorithms and Computation(알고리즘과 계산에 관한 일한 공동 워크숍)과 Korea-Japan Joint Workshop on Algorithms and Computation(알고리즘과 계산에 관한 한일 공동 워크숍)을 매년 번갈아가며 사용하기로요. 다만 약어에서는 통일성이 중요하니 두 나라 이름을 다 빼버리고 WAAC를 유지하기로 했습니다.


Figure 6.2: 2002 피파 월드컵 공식 로고

알파벳 순서를 따르는 방법도 있습니다. 그러나 2002 피파 월드컵에서 드러난 것처럼, 문제는 그리 단순하지 않습니다. 월드컵의 주관 단체인 피파는(FIFA, Fédération Internationale de Football Association) 1904년 프랑스에서 설립되었습니다. 한국은 프랑스어로 *Corée* 입니다. 일본은 프랑스어로 *Japon* 입니다. 프랑스어 알파벳에서는 *C*가 *F*보다 앞서 옵니다. 그래서 공식 로고에는 *Korea Japan* 이라고 쓰이게 되었습니다. 일본 사람들은 매우 불행해했습니다.

세계 협정시의 약자는 *UTC*입니다. 세계 협정시는 영어로 *Coordinated Universal Time*, 프랑스어로 *temps universel coordonné* 입니다. 영어권 사람들은 약자로 *CUT*를 제안했고, 프랑스어권 사람들은 *TUC*를 제안했습니다. 결국 아무 약자도 아닌 *UTC*로 타협했습니다.

순서를 정하기 위해 어떤 문자 체계를 써야 할까요? 위 두 예시에서 드러나듯, 영어 알파벳 순서는 유일한 해답이 아닙니다.

북한에서는 *북남*관계라고 말합니다. 남한에서는 *남북*관계라고 말합니다. 남한 사람들이 북한과 미국의 관계에 대해 말할 때는 대개 *북미*관계라고 합니다. 일부는 이러한 관행은 잘못되었다고 주장합니다. 미국은 자유민주주의의 수호자로서 남한을 경제적 및 군사적으로 지원해주었으나, 북한은 독재, 벼랑 끝 전술, 핵 개발 등 악행을 일삼아왔다는 것이 그들의 논지입니다. 이러한 주장은 "좋은 것 혹은 중요한 것이 문자적으로 앞서야 한다"는 가정에 기반합니다.

감사의 글 작성에는 불문율이 있습니다. 실질적으로 도움을 준 동료들과 편집자들의 이름은 앞부분에 옵니다. 그러나 가족, 친척, 친구, 배우자, 고양이, 개의 이름은 뒷부분에 옵니다. 저는 지금까지 어떤 예외도 보지 못하였습니다. 이 불문율은 가족이 가장 안 중요하다고 말해주는 걸까요? 제 가족은 수학이나 컴퓨터과학 지식이 없습니다. (가족들을 비난하는 것이 아닙니다.) 제 가족들이 무엇인가 과학에 의미 있는 공헌을 하기는 어려워 보입니다. (가족들을 비난하는 것이 아닙니다.) 이것이 바로 그들이 감사의 글 가장 뒷부분에 오는 이유일까요? 그러나 다른 과학자들의 가족은 제 가족과 다를지도 모릅니다. 그들의 가족은 과학을 알지도 모릅니다.

제가 무엇인가 크게 오해하는 걸까요? 그들이 맨 뒤에 오는 이유는 가장 안 중요해서가 아니라, 가장 중요해서이기 때문일 수 있습니다. 영웅은 언제나 최후에 등장하는 법입니다. 그들은 정신적 충족감의 원천입니다. 이는 삶에 있어서 과학에 비할 수 없이 중요합니다. 정신적으로 불안정한 상태로 좋은 연구를 하기란 불가능합니다. 안정성은 삶과 죽음의 문제입니다.

이름의 순서는 간과할 수 없는 문제입니다. 우리 모두에게 그렇습니다. 임의로 순서를 정한다면 누구도 만족하지 못합니다. 사람은 사회적 동물입니다. 명성과 평판은 인간 삶의 거의 전부입니다. 인간 뇌는 그렇게 되도록 진화했습니다. 누가 무엇을 받고, 누가 무엇을 받을 자격이 있는가, 이는 인간 서사의 영원한 주제입니다.

WAAC의 주최측에서는 매년 번갈아가면서 순서를 바꾸기로 했습니다. 원칙적으로는 저도 똑같이 할

수 있습니다. 저는 제 지인들께 본 학위논문을 나눠드리려고 계획중입니다. 다른 분들을 위해 각각 다른 판본을 준비할수도 있습니다. 그 분의 이름이 맨 앞에 오는 (혹은 맨 뒤에 오는) 바로 그 분을 위한 판본을 말이지요. 본인의 이름이 맨 앞에 오는 감사의 글을 읽고 모두가 기뻐할 것입니다. 저만 빼고. 여러분들은 그저 기뻐할지 모르지만, 저는 모든 감사의 글을 따로따로 하나하나 작업해야 합니다. 그렇게 되면 제 일이 너무 많아집니다. 논문 인쇄비용 또한 커집니다.

저는 이기적인 결정을 내렸습니다. 단 하나의 판본만을 준비할 것이며, 감사의 글에는 누구의 이름도 없을 것입니다. WAAC의 주최측도 학술대회명의 약자를 정할 때 똑같이 했습니다. 그 누구도 저를 비난할 수 없습니다. 저도 알고 있습니다. 이는 피장파장의 오류라는 것을. 다른 사람을 비난함으로써 제 악행을 정당화하려 한다는 것을. 저는 그저 순서를 결정하지 못하는 겁쟁이에 지나지 않습니다. 그리고 그 겁쟁이는 결정하지 않겠다는 결정을 내렸습니다. 결정하지 않겠다는 결정 또한 하나의 결정인 걸까요? 만약 그렇다면, 결국 결정을 내렸으니 겁쟁이가 아니라고 할 수 있겠네요. 어찌됐든 *겁쟁이*란 공허한 수사에 지나지 않습니다.

한심한 결정을 내려서 정말 죄송합니다. 감사의 글에 본인의 이름을 기대하는 분들께 정말 죄송합니다. 그러나 저는 충분히 제 입장을 설명했다고 생각합니다. 누가 끝까지 다 읽을지 의문인, 이 긴 감사의 글을 통해서요. 저는 그저 이름의 순서 때문에 여러분들을 화나게 만들고 싶지 않았을 뿐입니다. 선한 의도가 언제나 선한 결과를 낳지 않는다는 것 정도는 저도 압니다. 비록 그렇지만, 이 모든 것은 선한 의도에서 비롯되었다는 사실을 알아주셨으면 합니다. 정상을 참작해주셨으면 합니다.

여러분의 인내와 이해에 감사드립니다. 정말 고맙습니다. 쓸데없는 변명이 길었습니다. 이제부터 본격적으로 감사를 표현하고자 합니다.

여러분(일부 사망)들의 학문적인 도움 및 감정적인 도움 모두에 감사드립니다. 여러분(일부 사망)들의 진심어린 도움 및 별로 안 진심어린 도움 모두에 감사드립니다. (별로 안 진심어린 태도가 좋은 결과를 낳기도 합니다. 예를 하나 들어보겠습니다. 리처드 파인만은 노벨상을 받은 물리학자입니다. 그는 스스로가 재미있어서 연구했을 뿐, 인류를 위해 연구하지 않았습니다. 다른 예로는 시장 참가자들의 이기심이 있습니다. 각 참가자들은 스스로의 이익을 위해 일합니다. 자유시장경제체계하에서 각자의 이기심은 보이지 않는 손을 통해 경제 전체의 성장으로 종합됩니다. 사회주의 경제학자들은 이러한 예시에 동의하지 않겠지만 말이죠.) 여러분(일부 사망)의 도움이 없었다면 본 학위논문은 세상에 나오지 못했을 것입니다. 시각을 조금 바꿔보겠습니다. 여러분(일부 사망)의 악행이 이 한심한 출판물을 세상에 내놓았습니다. 본 학위논문이 과학적 공헌으로 인정받기는 어려울 것입니다. 정보의 홍수에 대한 공헌이라면 모를까.

일부 저자는 (굉장히 대단한 작품 집필을 끝마친 후) 감사의 글에서 주장합니다. “본서에서 잘 된 부분이 있다면 모두 여러분들이 도와주신 덕분입니다. 잘못된 부분은 모두 제가 부족한 탓입니다.” 이는 논리적 오류라고 저는 생각합니다. 두 마리 토끼를 모두 잡을 수는 없습니다.

저는 딜레마에 빠졌습니다. 두 가지 선택지가 있습니다. 본 학위논문의 미덕과 악덕을 모두 제 탓으로 돌리거나, 아니면 여러분(일부 사망) 탓으로 돌려야 합니다. 전자의 경우 감사 자체 의미없어집니다. 후자의 경우 두 가지 해석이 가능합니다.

- 여러분(일부 사망)의 도움은 대단한 작품을 낳았다.
- 여러분(일부 사망)의 도움은 한심한 작품을 낳았다.

한 이론컴퓨터과학 교수는 말했습니다. “제 석사학위논문, 지금 생각하면 부끄럽네요.” 부끄러워할 일이라고 합니다. 참고로 그는 제 지도교수입니다. 그의 석사논문과 달리, 제 석사학위논문은 대단한 작품이 될 수 있을까요? 저는 말했습니다. 본 논문이 과학적 공헌으로 인정받기는 불가능합니다. 만약 인정받는다면 정보의 홍수에 대한 공헌으로서의 인정뿐입니다. 학위논문 심사위원들을 제외하면 아무도 본 논문을 읽지 않을 것입니다. 두 해석 중 첫째 해석은 터무니없습니다.

작품의 질은 원인 귀속의 효과를 좌우합니다. 걸작을 창조했다면 안심해도 됩니다. 작품 탄생의 원인을 남에게 귀속해도 누구도 모욕으로 받아들이지 않습니다. 제 경우, 같은 처신이 부적절한 결과를 낳을 수

있음을 인지해야만 합니다.

한 가지 더 말씀드릴 내용이 있습니다. 제 석사학위논문에서, 아주 작은 부분만이 저의 독창적 생각에서 기인합니다. 저는 새로운 것을 발견하지는 못했습니다. 대부분은 이미 이전에 있던 것입니다. 즉, 제가 본 학위논문을 비판한다면, 이미 이전에 있던 것들을 비판하는 것이 됩니다. 이는 제 의도가 아닙니다. 상황은 더더욱 복잡해졌습니다. 제 한계를 벗어났습니다. 복잡성이란 절대 제가 다룰 수 없는 것입니다.

감사를 표현하려고 했습니다. 그러나 효과적인 방법을 찾지 못해 악전고투하고 있습니다. 이는 모두 제 사회성 부족 탓입니다. 비록 선한 결과로 이어질지는 알 수 없더라도, 다시 한 번 선한 의도로 퉁칠수는 없을까요? 알고 있습니다. 이는 감정에 호소하는 오류라는 사실을. 또 비형식적 오류를 범했습니다.

여러분(사망자를 포함하되 이에 국한되지 않음) 모두 고맙습니다. 제가 여기까지 올 수 있었던 것을 여러분(사망자를 포함하되 이에 국한되지 않음) 탓(덕분)이라고 말해야 할지는 잘 모르겠습니다. 이는 상술한 딜레마에 잘 드러나 있습니다. 여러분(사망자를 포함하되 이에 국한되지 않음)의 농담에 웃을 수 있었습니다, 그것이 농담으로서 의도되었는지는 잘 모르겠지만. 여러분(사망자를 포함하되 이에 국한되지 않음)의 격려와 보살핌에 감사드립니다.

# Curriculum Vitae

Name : Donghyun LIM (임동현)
E-mail : klimdhn@kaist.ac.kr
Date of Birth : April , 20th Century
Birthplace : 「私の心の故郷はインタネットです」— 孫正義, answering the question,「心の故郷はどこか」
Address : Earth, Solar System, Milky Way

## Educations

| | |
|---|---|
| 21st Century | Graduation from Elementary School |
| 21st Century | Graduation from Middle School |
| 21st Century | High School Diploma Qualification (고졸검정고시) |
| 21st Century | Graduation from Undergraduate Program |
| 21st Century | Hope to Graduate from Master Program (at the Moment) |
| 21st Century | Hope to Graduate from Doctoral Program (at the Moment) |

## Career

| | |
|---|---|
| 21st Century | Got LG Optimus Q2 (Smartphone) |
| 21st Century | Got Samsung Galaxy S4 (Smartphone, Secondhand) |
| 21st Century | Got Samsung Galaxy S8 (Smartphone, Secondhand) |
| 21st Century | No Driver's License Yet; Waiting for Autonomous Cars (at the Moment) |
| 21st Century | Death (Nobody Has Ever Avoided) |
| 22nd Century | Any Further? |

## Publications

1. Hope to publish a lot and get famous.
2. Martin told me dying my hair red and go singing would constitute a more feasible way for the objective of achieving fame. Figure 6.3 illustrates.
3. One common misinterpretation of Figure 6.3 is that people in some profession do not deserve the fame they have since they do not possess as much skill.
4. The correct interpretation is that *you will not become famous by doing science.*
5. At least my brother knows much more baseball players and YouTubers than scientists.
6. Why is the curve of athletes quadratic?

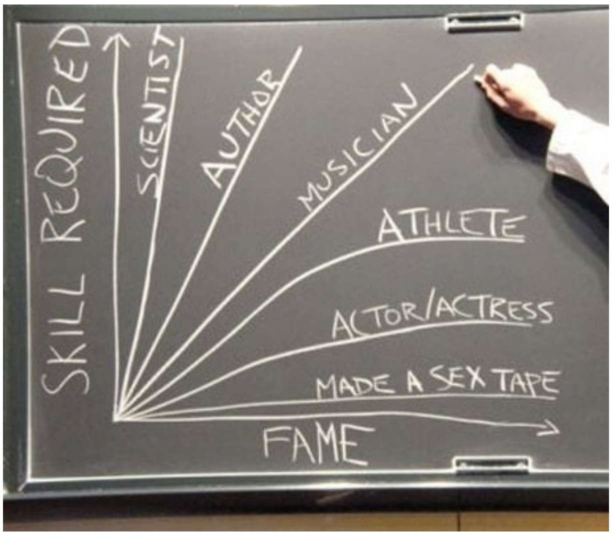


Figure 6.3: Relationship between Fame and Skill

7. My opinion is that still there are many ways to get famous by doing science. Proving or disproving $\mathcal{P} = \mathcal{NP}$ is one guaranteed road. Refusing Turing award or Fields medal is worth trying as well.
8. Scientists work indoors, and get more than average salary as far as I know. Science is good.
9. Some scientists work outdoors. Some perform dangerous experiments. Fortunately I am not one of them. I just stay in front of my computer for work. How easy. It tires my eyes and neck, though.
10. Many graduate students fail to earn minimum wage. Sewon argued that comfort of work cancels out low salary. He said he would not work at a convenience store even though that way he could earn minimum wage. He said it is because of the comfort of his current work in comparison to the difficult work at convenience store.
11. I believe in $\mathcal{P} = \mathcal{NP}$. The proof is simple and elementary. Canceling $\mathcal{P}$ on both sides yields

$$1 = \mathcal{N}.$$

1 is the identity with respect to multiplication. So we trivially have

$$\mathcal{P} = \mathcal{NP}.$$

12. Disclaimer: This proof is not original to me.
13. I know this proof is not rigorous. I am not sure whether I should call this a proof.
14. Calculus had been developed without rigorous foundation on real numbers or continuity. Intermediate value theorem, for example, was just believed to be true and invoked without justification. Rigorous proofs came up much later.
15. In the same manner, I believe, the informal proof of $\mathcal{P} = \mathcal{NP}$ can be turned into a formal one. I hope someday I would be able to contribute to bridging the logical gaps in the proof.
16. One way of making contribution is to develop an artificial superintelligence capable of doing mathematics. Machine learning seems one promising way toward the goal. It also seems one promising way toward a job, at least for the moment.